Dynamic Trading with Realization Utility*

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Abstract

An investor receives utility bursts from realizing gains and losses at the individual-stock level (Barberis and Xiong, 2009, 2012; Ingersoll and Jin, 2013) and dynamically allocates his mental budget between risky and risk-free assets at the trading-account level. Using savings, he reduces his stockholdings and is more willing to realize losses. Using leverage, he increases his stockholdings beyond his mental budget and is more reluctant to realize losses. While leverage strengthens the disposition effect, introducing leverage constraints mitigates it. Our model predicts that investors with stocks in deep losses sell them either immediately or after stocks rebound a little.

Keywords: disposition effect; mental account; deep losses; leverage; savings

JEL Classification: D03, G11, G12

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1 Introduction

One of the most robust findings about individual investors' trading behaviors is the disposition effect (Shefrin and Statman, 1985): An investor has a greater propensity to sell a stock that has gone up in value since purchase than one that has gone down.[1] Prospect theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992), in which preferences are defined over gains and losses, is often perceived as a natural explanation for the disposition effect. Barberis and Xiong (2009), henceforth BX (2009), show that the realization-utility implementation of prospect theory, where the investor receives a utility burst from realized gains and losses, more reliably predicts a disposition effect.

Barberis and Xiong (2012) and Ingersoll and Jin (2013), henceforth BX (2012) and IJ (2013), respectively, develop tractable intertemporal realization-utility models, significantly advancing our understanding of realization-utility investors' trading behaviors. We build on the powerful frameworks developed in BX (2012) and IJ (2013) but remove a key assumption in their papers: the investor has to allocate his entire stock-sale proceeds to the new stock that he buys. In practice, the investor does not restrict his trading strategies this way. More importantly, we show that once we lift this trading restriction, the model's predictions significantly change. To explain why it is crucial for us to remove this assumption, first consider the following commonly shared gambling experience.

A gambler allocates a budget Π_0 and leaves his credit cards home before going to casinos. He starts his first bet by choosing a fraction of Π_0. If he wins by G dollars ($G > 0$ for a win and $G < 0$ for a loss), he receives a utility burst that depends on G and then closes this betting episode. Afterwards, his budget for subsequent gambles on this trip changes to $\Pi_1 = \Pi_0 + G$ and the size of his next bet naturally depends on Π_1. This betting process continues until the end of his trip.[2] This narrative suggests that in addition to the series of mental accounts for each individual bet, the gambler also has a mental account for his entire trip with a stochastically evolving budget: $\{\Pi_n; n = 0, 1, 2, \cdots\}$. Next, we adapt this gambling narrative to our trading model.

[1] For evidence on disposition effects, see Odean (1998) for retail investors' stock trading, Heath et al. (1999) for executive stock option exercising, and Genesove and Mayer (2001) for the housing market.

[2] In an experimental setting with MBA students being the subjects and playing with real money, Thaler and Johnson (1990) study how prior outcomes affect risk choices. One of their findings is that a prior gain can stimulate risk seeking: the 'house money' effect. Thaler (1999) writes: "If a series of gambles are bracketed together then the outcome of one gamble can affect the choices made later" in the "Choice bracketing and dynamic mental accounting" section. Read, Loewenstein, and Rabin (1999) introduce "choice bracketing" to designate the grouping of individual choices together into sets and discuss narrow versus broad bracketing. See Kahneman and Lovallo (1993), among others, for related research.

An investor has a mental account with an initial budget of Π_0 for his *intertemporal* realization-utility optimization purpose. (This account is mentally separate from his other accounts, e.g., those for consumption smoothing and retirement purposes.) The investor opens his first investment episode when he buys a stock with a fraction of Π_0 and saves the unused budget in the risk-free asset. When selling this stock at time t, he receives a utility burst from the realized gains/losses (G_t) and closes this investment episode. He then obtains his new budget Π_t by combining his stock-sale proceeds with his savings. Afterwards, he starts his next investment episode by choosing a new stock with a new allocation (out of his new budget Π_t) and saves the unused budget.

In sum, the investor has two layers of mental accounts that are interconnected but serve different purposes. For each investment episode, he uses a stock-level mental account to evaluate his utility burst, based on the realized gains/losses. Additionally, he has a dynamic mental account, which brackets together all the individual-stock investment episodes via a stochastically evolving budget $\{\Pi_t; t \geq 0\}$ at the trading-account level. This mental account is broader than the stock-level mental account and is analogous to the gambler's mental account for his casino trip. Our two-layer mental accounts are consistent with An et al. (forthcoming) who find that investors have multiple mental accounts, e.g., at individual-stock and brokerage-account levels, and these mental accounts interact.

The interaction between the two layers of mental accounts suggests that both the *extensive margin* (when to open the next investment episode and which stock to buy), analyzed in the literature, and the *intensive margin* (how big the next stock position is) are important to the investor. At each time t, the investor can choose to save a fraction of his budget Π_t or use leverage when buying a new stock. By saving, the investor spreads out his stock trades over time, reducing downside risk and mitigating the disposition effect. Alternatively, by using leverage the investor increases his trading size beyond his budget.

The other key new feature of our model is that stock prices are discontinuous with downward jumps, as commonly observed in markets. Incorporating jumps not only makes the stock-price processes more realistic but more importantly generates new predictions on trading strategies, e.g., for stocks in deep losses, due to the interaction between the two new features of our model: jumps and the dynamic mental account.

Our model features three state variables: risk-free wealth W_t, risky wealth X_t, and the reference level B_t, which we use as the base to calculate realized gains and losses, at time t. Using the homogeneity property, we solve the model by working with 1.) scaled risky wealth $x_t = X_t/B_t$ as in BX (2012) and IJ (2013); and 2.) scaled risk-free wealth $w_t = W_t/B_t$, which is new in our model. Note that the investor's risky wealth and risk-free savings are not completely fungible due to his two-layered mental accounts. Next, we discuss our model's key predictions.

The first set of predictions centers around the investor's *intensive margin*, i.e., the allocation of his budget between a new stock he chooses and the risk-free asset, and the effect of this intensive margin on his trading behavior.

Saving and Leverage (Intensive Margin) and Implications. When trading stocks at t, the investor allocates a fraction of his budget Π_t to the risk-free asset and the remaining to a stock by targeting a constant ratio between his risk-free wealth and risky wealth, w^* (after netting out transaction costs). By saving ($w^* > 0$), the investor makes smaller trades and spreads out his trades over time, lowers transaction costs (in dollars), and becomes less sensitive to the loss in his stock position. Because only a fraction of his budget is at risk, he realizes both gains and losses more frequently. In contrast, by using leverage ($w^* < 0$), the investor increases his trading size beyond his budget. Because of increased risk exposures and larger transaction costs (in dollars), the investor is more reluctant to realize losses, strengthening the disposition effect. In contrast, the option to save and use leverage has a negligible effect on the investor's gain realizations. This is because his preference over gains is concave in the gain region making him want to lock in gains.

Our model's predictions on gain- and loss-realizations are consistent with findings in Barber et al. (2019) and Heimer and Imas (2022). According to Heimer and Imas (2022), "Access to leverage increased the disposition effect; ... this increase was driven by a greater propensity to hold losses; gains were realized at the same rate" (p. 1646). Barber et al. (2019) find that "margin investors have a stronger disposition effect than cash investors" (p. 4).

Determining Optimal Target: w^*. How does an investor determine the optimal mix between the risk-free asset and the stock he chooses (w^*) when opening a new investment episode? Consider the effect of stock return volatility. For volatile stocks, the investor saves a fraction of his budget Π into the risk-free asset ($w^* > 0$) to reduce his trading account's (dollar) risk exposure and trades more frequently. In contrast, for stocks with low volatility, the investor uses leverage to increase his trading size beyond his budget

($w^* < 0$) and trades less frequently. As a result, an investor using leverage is subject to a stronger disposition effect, *ceteris paribus* (Heimer and Imas, 2022).

Our model predicts that investors prefer stocks with either high or low volatility over stocks with intermediate volatility. This is because the value of saving in the risk-free asset is high for stocks with high volatility and the value of leverage is high for stocks with low volatility, but the value of saving and leverage is low for stocks with intermediate volatility. In sum, the flexibility of using the risk-free asset either via saving or leverage can be quite valuable for investors in either a high- or low-volatility environment.[3]

Leverage Constraints. So far we have analyzed the effect of leverage for the case where leverage constraints do not bind. What if they do? By forcing the investor to realize losses sooner than he would prefer (so that creditors break even), binding leverage constraints makes the investor realize losses sooner, mitigating his disposition effect. Anticipating this contingency, the investor lowers his leverage *ex ante*, which in turn makes him more risk tolerant. Our model's prediction is in line with Heimer and Imas (2022) who find that leverage constraints can improve financial decision-making by mitigating the disposition effect.

Next, we characterize our model solution via two cases (a and b) that differ in the number of solution regions, which in turn generate different time-series predictions.

Two Cases of Model Solution: (a.) the three-region case and (b.) the four-region case. For both cases a and b, there are a gain-realization region where $x \geq \overline{x}^*$ and a normal holding region where $x \in (\underline{x}^*, \overline{x}^*)$. Whenever x exceeds the endogenous gain-realization boundary $\overline{x}^*$, the investor voluntarily realizes gains. If gains ($1 < x < \overline{x}^*$) or losses ($\underline{x}^* < x < 1$) are small or moderate, where $\underline{x}^*$ is the endogenous loss-realization boundary, the investor does not trade. The preceding two predictions are the same as in the literature. Our model's new predictions come from the region(s) to the left of $\underline{x}^*$.

- ***a.*** In the three-region case, the third region where $x \in (0, \underline{x}^*)$ is the loss-realization region where the investor voluntarily realizes losses for all $x \in (0, \underline{x}^*)$.
- ***b.*** In the four-region case, there are two regions to the left of $\underline{x}^*$. One is the loss-realization region where $x \in (\underline{x}^{**}, \underline{x}^*)$ with $\underline{x}^{**}$ and $\underline{x}^*$ being the lower and upper

[3] Bian et al. (2021) show that stocks bought in margin accounts tend to have lower systematic volatility and total volatility than stocks bought in cash accounts, consistent with our model's prediction.

boundaries of this region, respectively. The other region where $x \in (0, \underline{x}^{**})$ is the *deep-loss holding* region, which is absent in case a.

A key difference between the two cases is whether the investor wants to sell his stock in deep losses. A stock position is in deep losses, if its value X_t is significantly below its reference level B_t, i.e., if $x_t = X_t/B_t$ is close to zero. We show that whether the investor voluntarily sells his stock in deep losses critically depends on whether he has set aside sufficient savings (w^*) in his trading account. Below, we further discuss how this (intensive) savings margin generates different time-series predictions.

Selling a Stock in Deep Losses: Case a. Why would a realization-utility investor ever want to sell a stock in deep losses in our model? This is because the investor has set aside sufficient savings in the risk-free asset. Consider the following example. Andrew with a mental budget of \$100 in his account allocates \$10, which is 10% of his budget (\$100), and later loses 90% on this stock. Realizing this stock's deep loss would leave him with a budget of $\$91 = \$10 \times (1 - 90\%) + \$90$ in his mental account, implying a 9% loss of his mental budget of \$100 in his trading account. With a mental budget of \$91, the present value of utility bursts from his future trading activities is larger than the immediate utility cost of realizing this stock's deep loss. This is why Andrew is willing to voluntarily sell his stock in deep losses.

Voluntarily selling a stock in deep losses is a unique prediction of our model. This prediction is broadly consistent with the portfolio-driven disposition effect (An et al., forthcoming): the disposition effect is large when the portfolio is at a loss but nearly disappears when the portfolio is at a gain. To voluntarily sell stocks in deep losses, it is necessary for the model to have (downward) asset-price jumps and (sufficiently) large savings the risk-free asset, both of which are new features of our model. Without jumps, the deep-loss region cannot be reached on the optimal path. Without large savings, it is always optimal for the investor to hold onto his stock positions in the deep-loss region.

What if the investor's savings (w^*) are small? This is our case b.

Selling a Stock after It Rebounds: Case b. Consider the following example. Brian with a mental budget of \$100 allocates \$80 and later loses 90% on his stock. Realizing his stock's deep loss is just too painful because doing so would leave him with a budget of only $\$28 = \$80 \times (1 - 90\%) + \$20$, mapping to a huge (72%) loss of his trading budget \$100. However, if the stock rebounds, cutting his stock loss from 90% to 75%, Brian's budget would bound back to $\$40 = \$28 + \$80 \times 15\%$. This stock rebound reduces the loss of his mental budget by just enough ($\$12 = \$40 - \$28$ in our example) so that his utility cost of realizing losses is dominated by the benefit of doing so, which is to reset his reference level for future trading.

This prediction is consistent with our observation that retail investors often sell their losing stocks after these stocks rebound a bit, which cannot be generated by diffusion models with standard reference-point dynamics, i.e., those with constant growth rates.[4]

In terms of solution regions, the above example suggests: (1.) there are two disconnected holding regions: the normal and the deep-loss holding regions, and (2.) between the two holding regions is the loss-realization region. In sum, when the investor's savings are not sufficiently large, the solution features four regions: a gain-realization region, a loss-realization region, and the two disconnected holding regions.

Related Literature. BX (2009) analyze two implementations of prospect theory, one based on realized gains/losses and the other based on annual gains/losses, and find that the former more consistently predicts the disposition effect.[5]

Table 1: Comparing Our Model with BX (2012), IJ (2013), and HY (2019).

	Saving or Leverage	Leverage Constraint	Jump Shocks	Number of Regions	All Regions Reachable?	Realize Deep Losses?	Realize Losses after Rebound?
BX (2012)	No	N/A	No	2	Yes	No	No
IJ (2013)	No	N/A	No	4*	No	No	No
HY (2019)	No	N/A	No	4*	No	No	No
Our Model	Yes	Yes	Yes	4*	**Yes**	No	**Yes**
				3	**Yes**	**Yes**	No

This table compares the (infinite horizon with no liquidity shocks) models in the four papers.
*Models feature four-region solutions can produce two-region solutions for certain parameter values.

BX (2012) show that an investor with piecewise linear realization utility (and loss aversion larger than one) realizes gains when the stock he owns goes up by a certain percentage but never voluntarily realizes losses as the utility cost from realizing losses is

4 Models of BX (2012) and IJ (2013), when extended to a specification where the reference point is an average of past stock prices (with an exponentially decaying weights), can also generate this prediction. Alternatively, a belief-based model with the law of small numbers (Rabin and Vayanos, 2010) can also generate this prediction. We thank a referee for providing these alternative explanations.

5 Kyle, Ou-Yang, and Xiong (2006) analyze one-time liquidation problems for a decision maker with prospect theory preferences but with no reinvestment options. Li and Yang (2013) develop a general-equilibrium model to examine the asset-pricing and trading-volume implications of prospect theory via the disposition effect.

too high compared with the benefit of resetting the reference for gain realizations in the future. As a result, there are two regions, a gain-realization region where $x \geq \overline{x}^*$ and a holding region where $x \in (0, \overline{x}^*)$ in BX (2012). We show that the BX (2012) result of no voluntary loss realization for investors with piecewise linear utility continues to hold in our model when he can save a fraction of his budget or use leverage. Additionally, the piecewise-linear-utility investor may use leverage but never saves: $w^* \leq 0$.

IJ (2013) incorporate S-shaped utility into the realization-utility framework proposed by BX (2009, 2012). A key prediction of IJ (2013) is that the investor is willing to voluntarily realize losses in order to reset his reference level for future gain realizations. This is because the investor in IJ (2013) becomes less sensitive to losses as his losses increase unlike in BX (2012). Specifically, IJ (2013) analyze the investor's gain- and loss-realization strategies by characterizing the two cutoff thresholds that define the normal holding region along the optimal path.

He and Yang (2019), henceforth HY (2019), extend IJ (2013) to allow for a general S-shaped realization utility, a terminal expected utility, and an adaptive reference point. HY (2019) show that the solution for diffusion realization-utility models features *four* regions in general, with a deep-loss holding region being the fourth. To the best of our knowledge, HY (2019) is the first to discuss all four regions and point out that the deep-loss holding region is not reached on the optimal path in diffusion models.[6] Note that diffusion models with S-shaped utility may only feature two regions for some parameter values, because realizing losses is just too painful as in BX (2012).[7]

We summarize the differences between our model and the three closely related papers discussed above in Table 1. This table's first column shows that the investor in our model either saves or uses leverage, different from the other papers. This is because the investor in our model has two mental accounts: a narrower mental account at the stock level for utility-burst calculations as in the other papers, but also a broader mental account for his intertemporal trading. It is this broader mental account that allows him to separate his stock position from the budget of his intertemporal mental account.

Our model allows us to introduce a leverage constraint which mitigates the disposition effect by forcing the investor to realize losses (see the second column). The third column

[6] While implicitly containing four regions, IJ (2013) do not mention the deep-loss holding region. Because this fourth region is not on the optimal path, it is sufficient to use the other three regions to fully characterize the solution on the optimal path as in IJ (2013).

[7] For example, when the S-shaped utility is sufficiently close to the piecewise linear utility as in BX (2012), the solution then only has two regions. In this special two-region case, the upper boundary for the loss-realization region $\underline{x}^*$ equals zero. IJ (2013) provide more discussions, e.g., their Proposition 1.

highlights jumps, another key feature of our model. We show that only with jumps can all the regions (up to four) be visited on the optimal path in our model.

The fourth column of Table 1 shows that our model solution features two mutually exclusive cases: case a with three regions and case b with four regions. In contrast, the solution in the other three models features one case. The fifth column shows another key difference across these models. All the regions are reachable on the optimal path due to jumps in our model while the fourth (deep-loss) region in IJ (2013) and HY (2019) is not reachable.

The sixth column shows that case a of our model uniquely predicts voluntary stock deep-loss realizations. The mechanism generating this result again goes back to the investor's dynamic mental account. With sufficient savings, the investor has incentives to sell his stock even in deep losses to reset the reference level for his future trading.[8]

The seventh column of Table 1 shows that case b of our model predicts the following time series: While an investor is unwilling to sell a stock in deep losses, he voluntarily realizes losses after the stock price rebounds a bit. This is because this rebound cuts his losses by just enough so that his benefit of realizing losses to reset his reference level for future trading is larger than the cost of realizing these losses.

One way to highlight a key difference across the four models is as follows. While generating very similar gain-realization strategies (i.e., to lock in small gains), these four models predict very different loss-realization strategies. In BX (2012), investors do not voluntarily realize losses. In IJ (2013) and HY (2019), investors are willing to realize moderate losses but not deep losses. In our model, investors voluntarily realize both moderate and deep losses.

2 Model

An investor has an account with a budget of $\Pi_0 > 0$ at $t = 0$ solely for the purpose of his intertemporal realization-utility optimization. He mentally separates this account from his other accounts, e.g., those for consumption smoothing and retirement purposes. The investor only receives a utility burst from realized gains and losses of the stock sale. While the investor has a stock-level mental account for each utility-burst calculation, he also has a broader mental account for his intertemporal realization utility at the

[8] In contrast, in the three papers discussed above, the investor does not voluntarily realize deep losses because his entire trading account holds a single stock. Realizing deep losses would effectively wipe out his mental budget, which yields a high utility cost but a negligible benefit for future gain realizations.

trading-account level. In our two-layered mental-account model, the investor is not required to spend his entire budget Π_t when he trades. Instead he typically holds both a stock and the risk-free asset. In contrast, earlier realization-utility models, e.g., BX (2012), IJ (2013), and HY (2019), allow the investor to make a binary choice in that he either holds a stock he chooses or invests in the risk-free asset at any time.[9] In sum, an investor in our model has both an extensive margin (whether to invest in a stock) as in their models but also an intensive margin (the size of his stock position). Finally, his mental account is self financing.

2.1 Trading Opportunity: Multiple Stocks and Risk-free Asset

The risk-free asset pays interest at the constant rate of $r > 0$. There are multiple (N) risky assets (stocks) indexed by $n \in \{1, 2, \cdots, N\}$. The cum-dividend price process for a unit of asset n, $P_{n,t}$, follows a geometric Brownian motion (GBM) process:

$$dP_{n,t} = \mu P_{n,t} dt + \sigma P_{n,t} d\mathcal{Z}_{n,t} \,, \quad t > 0 \,, \tag{1}$$

where $\mathcal{Z}_{n,t}$ is a standard Brownian motion. Following the realization-utility literature, we assume that at each moment $t > 0$, the investor can hold at most one of the N stocks due to mental accounting.[10] When purchasing and selling a stock, the investor pays proportional transaction costs. Let θ_p and θ_s denote the proportional purchase and sale cost parameters, respectively. Finally, we set the expected return (drift) μ and volatility $\sigma > 0$ to be the same for all stocks as in this literature.

Let τ_i denote the investor's i-th stock trading time. Importantly, in our model, in addition to deciding the extensive margin, which stock to buy at what time τ_i, the investor also chooses the *intensive margin*, the trading size X_{τ_i+}. During an investment episode between two consecutive trading moments: (τ_i, τ_{i+1}), the investor's allocation to the stock in dollars (risky wealth), X_t, follows the same GBM process as $P_{n,t}$ does:

$$dX_t = \mu X_t dt + \sigma X_t d\mathcal{Z}_{n,t} \,, \quad t \in (\tau_i, \tau_{i+1}) \tag{2}$$

and the investor's risk-free wealth, W_t, evolves as:

$$dW_t = rW_t dt \,, \quad t \in (\tau_i, \tau_{i+1}). \tag{3}$$

[9] These papers naturally focus on the economically interesting case where the investor always holds a stock on the optimal path. See their papers for additional discussions on conditions under which the investor voluntarily holds a stock.

[10] Because the investor can hold at most one stock at any point in time, the correlation matrix of these N stock returns plays no role in our model as in this literature.

At any $t > 0$, the investor has an option to sell his stock holdings X_t and obtain:

$$\Pi_t = W_t + (1 - \theta_s) X_t \,, \tag{4}$$

where $\theta_s X_t$ is the cost of selling the stock.[11] We refer to Π_t as the budget at t for this mental trading account. Upon selling the stock he owns at τ_i, the investor then allocates his budget Π_{τ_i} between a new stock he chooses and the risk-free asset. When he does not trade, the budget given in (4) measures his mark-to-market wealth in this mental account under the counterfactual that he sells the stock he owns at time t. The investor's mental budget Π_t proves helpful when we discuss the economics of the investor's trading behaviors, even though it is not a state variable.

When buying a new stock with a (dollar) position of X_{τ_i+} at τ_i+, the investor pays from his mental account and therefore his post-trading risk-free wealth W_{τ_i+} is given by:

$$W_{\tau_i+} = \Pi_{\tau_i} - (1 + \theta_p) X_{\tau_i+} \,, \tag{5}$$

where $\theta_p X_{\tau_i+}$ is the purchasing cost. In prior realization-utility models, the investor is required to use his entire budget Π_{τ_i} when acquiring a new stock. This implies $X_{\tau_i+} = (1 - \theta_s) X_{\tau_i} / (1 + \theta_p)$ at trading time τ_i. Therefore, once τ_i is chosen, the trading size is determined because there are no savings: $W_t = 0$ at all t.

Leverage Constraint. In addition to saving a fraction of his budget in the risk-free asset, the investor can also borrow using the stock he buys as collateral, provided

$$X_t \geq -W_t / \kappa \,, \tag{6}$$

where the condition $0 < \kappa < 1 - \theta_s$ ensures that creditors bear no credit risk and hence are willing to lend at the risk-free rate.[12] When the leverage constraint (6) binds, the investor is forced to realize losses. If $\kappa = 0$, the investor is not allowed to borrow.

2.2 Realization Utility

We model the investor's preferences using the realization utility proposed by BX (2012) and IJ (2013). An investor views his investing process as a series of separate episodes and that his utility payoff comes in a burst when realizing gains or losses from his stock sale. To calculate utility from realized gains (losses), we need a reference level. As in BX

[11] For tractability, as in the realization-utility literature, we assume that the investor sells his entire stock position to close his current investment episode before acquiring a new stock.

[12] This is because $(1 - \theta_s) X_t + W_t \geq \kappa X_t + W_t \geq 0$, implied by $0 < \kappa < 1 - \theta_s$ and (6).

(2012), we assume that the reference level, $B_t > 0$, grows exponentially at the risk-free rate r between two consecutive trading moments, τ_i and τ_{i+1}:

$$dB_t = rB_t dt \quad \text{for} \;\; t \in (\tau_i, \tau_{i+1})\,. \tag{7}$$

After adjusting his stock holdings at τ_i+, the reference level B_{τ_i} is reset as follows:

$$B_{\tau_i+} = X_{\tau_i+}\,. \tag{8}$$

Let G_{τ_i} denote the realized gain (loss) at τ_i after closing the investment episode:

$$G_{\tau_i} = (1-\theta_s)X_{\tau_i} - B_{\tau_i}\,. \tag{9}$$

Anticipating that the homogeneity property plays an important role in our model solution, we define g_{τ_i} as the realized gain or loss, G_{τ_i}, scaled by the reference level B_{τ_i}:

$$g_{\tau_i} = G_{\tau_i}/B_{\tau_i}\,. \tag{10}$$

As in IJ (2013), the investor derives the following utility burst when selling the stock and realizing a gain or loss at the trading time τ_i:

$$U(G_{\tau_i}, B_{\tau_i}) = B_{\tau_i}^{\beta} u(G_{\tau_i}/B_{\tau_i}) = B_{\tau_i}^{\beta} u(g_{\tau_i})\ , \tag{11}$$

where $\beta \in (0,1]$ is a constant and $u(\,\cdot\,)$ is a function that depends on the scaled realized gain or loss, g_{τ_i}.[13] Next, we specify $u(\,\cdot\,)$ by adopting a reference-level-scaled version of the Cumulative Prospect Theory (CPT) utility of Tversky and Kahneman (1992):[14]

$$u(g) = \begin{cases} g^{\alpha_+} & \text{if } g \geq 0\,, \\ -\lambda(-g)^{\alpha_-} & \text{if } g < 0\,, \end{cases} \tag{12}$$

where $\lambda \geq 1$ and $\alpha_\pm \in (0,1]$ are the three constant parameters describing $u(\,\cdot\,)$.

Additionally, $u(\,\cdot\,)$ inherits two other features of CPT: (i) the diminishing sensitivity: $u(\,\cdot\,)$ is concave ($0 < \alpha_+ \leq 1$) in the gain ($g \geq 0$) region but convex ($0 < \alpha_- \leq 1$) in the loss ($g < 0$) region, and (ii) loss aversion ($\lambda \geq 1$). A higher value of λ refers to a stronger loss aversion. Note that the $g = 0$ point is a kink where $u(\,\cdot\,)$ is not differentiable. Finally, as in IJ (2013), we require:

$$\beta \leq \min\{\alpha_+, \alpha_-\}, \tag{13}$$

[13] The specification in (11) makes our model growth stationary and tractable in line with the finance tradition as noted by IJ (2013).

[14] The realization utility formulation of prospect theory ignores probability weighting, another key feature of CPT. Investors tend to overweight the tail outcomes of a probability distribution. Put differently, investors typically prefer lotteries and insurances compared with predictions of expected utility models.

which ensures that $|U(G,B)|$ is decreasing in B for a fixed G. When $\beta < \min\{\alpha_+, \alpha_-\}$, the smaller the reference level, the greater the utility impact of a realized gain or loss in the absolute value ($|G|$).[15] For the special piecewise linear realization utility case used in BX (2012) where $\beta = \alpha_+ = \alpha_- = 1$, $U(G,B)$ is independent of the level of B (for a given G) and does not feature the diminishing sensitivity property.

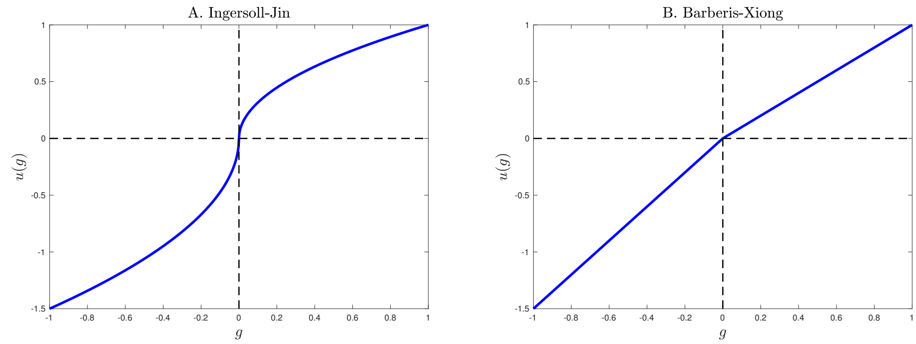

Figure 1: SCALED REALIZATION UTILITY $u(\cdot)$. Panel A plots the $\alpha_\pm = 0.5$ case in IJ (2013). Panel B plots the piecewise linear $\alpha_\pm = 1$ case in BX (2012). In both panels, the loss aversion parameter is $\lambda = 1.5$ and $u(\cdot)$ is not differentiable at the kink point $x = 0$. In Panel A, $u(\cdot)$ is convex in the loss region and concave in the gain region. In Panel B, $u(\cdot)$ is globally concave.

Figure 1 plots the scaled realization utility $u(\,\cdot\,)$. Panel A plots the $\alpha_\pm = 0.5$ case used in IJ (2013). Panel B plots the $\alpha_\pm = 1$ case analyzed in BX (2012). In both panels, $\lambda = 1.5$. In Panel A, $u(\,\cdot\,)$ is S-shaped: convex in the loss region and concave in the gain region. In contrast, for the piecewise linear case, $u(\,\cdot\,)$ is globally concave (Panel B). In both panels, $u(\,\cdot\,)$ is not differentiable at $g = 0$.

Liquidity Shocks. Following BX (2012), we assume that the investor faces an exogenous liquidity shock, which arrives stochastically at a constant rate of $\xi > 0$. Upon the arrival of this shock at τ_L, the investor immediately sells his entire stock holdings, realizes a burst utility given in (11) and exits from the asset market. As in BX (2012), we incorporate the liquidity shock for the following reasons. It captures a sudden consumption need that forces the investor to draw on the funds and also makes the investor care about paper gains and losses to some degree, which is reasonable. Finally, this shock allows us to calibrate the investor's expected trading horizon to $1/\xi$.

Investing in a Stock or Not? As in BX (2012), liquidity shocks force the investor to involuntarily sell the stock he owns. Anticipating this contingency, the investor may not

[15] Here is an illustrative example from IJ (2013): "the gain or loss of \$10 is felt more strongly when the reference level is \$100 than when it is \$500."

want to invest in a stock *ex ante*. Intuitively, if the upside of realizing gains is sufficiently large, the investor will choose to invest in a stock. Before solving for the binary decision at $t = 0$, we first characterize the investor's optimization problem conditional on his choosing to invest in a stock.

2.3 The Optimization Problem

For a given triple (W_t, X_t, B_t) at time t, the investor chooses a sequence of trading times $\{\tau_i \geq t;\ i = 1, 2, \cdots\}$ and allocation of X_{τ_i+} to a stock at each τ_i+ to solve

$$\max_{\{\tau_i,\, X_{\tau_i+}\}} \mathbb{E}_t \left[\sum_{i=1}^{\infty} e^{-\delta(\tau_i - t)} U(G_{\tau_i}, B_{\tau_i}) \mathbf{1}_{\{\tau_i < \tau_L\}} + e^{-\delta(\tau_L - t)} U(G_{\tau_L}, B_{\tau_L}) \right], \tag{14}$$

subject to the leverage constraint (6) and the dynamics described by (2), (3), (5), (7), and (8). In (14), $\delta > 0$ is the investor's subjective discount rate and $\mathbf{1}_A$ is the indicator function.[16] There are three state variables: risk-free wealth (W), risky wealth (X), and the reference level (B). Let $V(W, X, B)$ denote the value function for the problem defined in (14). Next, we characterize this problem recursively.

At the trading time $\tau \geq t$, the investor realizes a gain or loss, receives a direct utility burst of $U((1-\theta_s)X_\tau - B_\tau, B_\tau)$, and moves forward with a continuation value of:

$$\widehat{V}(\Pi_\tau) = \max_{X_{\tau+}} V(W_{\tau+}, X_{\tau+}, X_{\tau+}), \tag{15}$$

where $\Pi_\tau = W_\tau + (1-\theta_s)X_\tau$ is the post-realization budget given in (4). Equation (15) states that at $\tau+$ (before the liquidity shock arrives), the investor chooses $X_{\tau+}$ out of his budget Π_τ to maximize $V(W_{\tau+}, X_{\tau+}, X_{\tau+})$ subject to (5) and the leverage constraint (6). Let $F(W_\tau, X_\tau, B_\tau)$ denote the total (utility) payoff function when closing an episode:

$$F(W_\tau, X_\tau, B_\tau) = U((1-\theta_s)X_\tau - B_\tau, B_\tau) + \widehat{V}(\Pi_\tau). \tag{16}$$

In sum, we can express the optimization problem given in (14) as follows:

$$V(W_t, X_t, B_t) = \max_{\tau \geq t} \mathbb{E}_t \left[e^{-\delta(\tau - t)} F(W_\tau, X_\tau, B_\tau) \mathbf{1}_{\{\tau < \tau_L\}} + e^{-\delta(\tau_L - t)} U(G_{\tau_L}, B_{\tau_L}) \right], \tag{17}$$

subject to (2)–(8). Intuitively, the following HJB equation characterizes the investor's value function in the region where the investor does not trade:

$$\delta V = \frac{1}{2}\sigma^2 X^2 V_{XX} + \mu X V_X + rWV_W + rBV_B + \xi \left[U(G, B) - V \right]. \tag{18}$$

At trading time τ, $V(W_\tau, X_\tau, B_\tau) = F(W_\tau, X_\tau, B_\tau)$.

[16] $\mathbf{1}_A = 1$ if event A occurs and $\mathbf{1}_A = 0$ otherwise.

Voluntary Participation. Finally, to ensure that the investor voluntarily participates in investing in stocks, we require $\widehat{V} > 0$, where $\widehat{V}$ given in (15).

3 Solution

We solve the optimization problem given in (17) in two steps. First, conditioning on his voluntarily investing in a stock, we simplify his problem by using our model's homogeneity property to characterize the solution.[17] Second, we provide a condition under which the investor voluntarily invests a fraction of his budget in the stock.

Using the Homogeneity Property to Simplify the Problem (17). Let w_t and x_t denote W_t and X_t scaled by the contemporaneous reference level B_t for all t, respectively:

$$w_t = \frac{W_t}{B_t} \quad \text{and} \quad x_t = \frac{X_t}{B_t}\,. \tag{19}$$

Between two consecutive trading moments (τ_i, τ_{i+1}), the x_t process is a GBM:

$$dx_t = (\mu - r)x_t dt + \sigma x_t d\mathcal{Z}_{n,t}\,. \tag{20}$$

Since both the reference point B_t and the risk-free asset holdings W_t grow at the risk-free rate r, the scaled risk-free asset holdings w is constant over (τ_i, τ_{i+1}) and therefore

$$dw_t = 0\,. \tag{21}$$

The homogeneity property allows us to write $V(W,X,B)$ and $F(W,X,B)$ as $V(W,X,B) = B^\beta v(w,x)$ and $F(W,X,B) = B^\beta f(w,x)$ for any $B > 0$, where $v(w,x)$ and $f(w,x)$ are the scaled value and payoff functions to be characterized next.

Following BX (2012), we define $\delta_e = \delta - \beta r$ and interpret δ_e as the investor's effective discount rate. Using the homogeneity property, we can simplify the problem (17) as:

$$v(w_t, x_t) = \max_{\tau \geq t} \mathbb{E}_t \left[e^{-\delta_e(\tau - t)} f(w_\tau, x_\tau) \mathbf{1}_{\{\tau < \tau_L\}} + e^{-\delta_e(\tau_L - t)} u((1-\theta_s)x_{\tau_L} - 1) \right], \tag{22}$$

where $f(w_\tau, x_\tau)$ equals the sum of the utility burst and the continuation value:

$$f(w_\tau, x_\tau) = u((1-\theta_s)x_\tau - 1) + [(1-\theta_s)x_\tau + w_\tau]^\beta \cdot \widehat{v}\,. \tag{23}$$

The continuation value is homogeneous of degree β in budget, $\pi_\tau = (1-\theta_s)x_\tau + w_\tau$, and equals $\pi_\tau^\beta\, \widehat{v}$, where $\widehat{v}$ is the (utility) value with a budget of one dollar, solving:

$$\widehat{v} \; = \; \max_{w \geq -\kappa} \; m(w)\,, \tag{24}$$

[17] We solve the optimization problem using the variational inequality in Appendix A.

where $m(w)$ is the (utility) value with a given scaled risk-free wealth w:

$$m(w) = \left(\frac{1}{w+1+\theta_p}\right)^{\beta} v(w,1)\,. \tag{25}$$

An investor with a budget of $\Pi_\tau = 1$ at τ, targeting a ratio of $W_{\tau+}/X_{\tau+} = w$ at $\tau+$, must allocate $X_{\tau+} = \frac{1}{w+1+\theta_p}$ to the stock.[18] As $B_{\tau+} = X_{\tau+}$, we can rewrite (25) as $B^{\beta}_{\tau+} v(w,1)$, which is homogeneous in $B_{\tau+}$ with degree β, as expected. The investor chooses the optimal w^* at $\tau+$ to maximize his utility given in (24)-(25). In contrast, in the litearture, (24) and (25) are the same up to a proportionality constant $\left(\frac{1}{1+\theta_p}\right)^{\beta}$ because $w_t = 0$ for all t by assumption.

In sum, because the investor has a dynamic mental budget at the trading account level, he either saves a fraction of his budget in the risk-free asset or uses leverage by managing w_t subject to the leverage constraint (6). This is very different from the literature where $w_t = 0$ is assumed to hold at all t conditional on the investor's voluntary participation. We show that this new decision margin (w) fundamentally changes the trading strategies and value function. Next, we characterize the solution.

Characterizing the Solution Using the HJB Equation. At any t, the investor can either trade or passively hold onto his stock position. The solution thus features two types of *domains*. If the investor realizes gains or losses, his value function must equal the payoff function: $v(w,x) = f(w,x)$. We refer to the set of (w,x) on which the investor trades as the realization domain $\mathcal{R}$, where $v(w,x) = f(w,x)$.

If the investor chooses to hold onto his stock positions, doing so must yield a higher value than realizing gains (losses). We refer to this set of (w,x) as the holding domain $\mathcal{H}$ where $v(w,x) > f(w,x)$. The following HJB equation holds for $v(w,x)$ defined in (22):

$$\delta_e v(w,x) = \frac{1}{2}\sigma^2 x^2 v_{xx}(w,x) + (\mu - r)x v_x(w,x) + \xi\left[u((1-\theta_s)x - 1) - v(w,x)\right]. \tag{26}$$

Our model solution features two cases as summarized in Table 1: (a) the three-region case and (b) the four-region case. The three regions in case a are: a gain-realization region where $x \in (\overline{x}^*, \infty)$, a loss-realization region where $x \in (0, \underline{x}^*)$, and a normal holding region where $x \in (\underline{x}^*, \overline{x}^*)$ that lies between the two realization regions. The investor voluntarily sells his stock even in deep losses because he has set aside some savings for his future trading. This is a unique prediction of our model.

The four regions in case b are: a gain-realization region where $x \in (\overline{x}^*, \infty)$, a normal holding region where $x \in (\underline{x}^*, \overline{x}^*)$, a loss-realization region where $x \in (\underline{x}^{**}, \underline{x}^*)$, and a

[18] This is because adding up $W_{\tau+}$, $X_{\tau+}$, and the trading cost $\theta_p X_{\tau+}$ shall equal the budget $\Pi_\tau = 1$.

deep-loss holding region where $x \in (0, \underline{x}^{**})$.[19] We return to discuss the implications of these two cases in Section 5 where we analyze the effect of jumps. This is because the economically interesting deep-loss region is not reachable on the optimal path in diffusion models but is reachable in models with jumps.

Note that the leverage constraint (6) plays a prominent role in our analysis. When it binds ($w = -\kappa x$), the investor is forced to realize losses and $v(-\kappa x, x) = f(-\kappa x, x)$.

Finally, an investor is willing to invest in a stock if $\widehat{v} > 0$, where $\widehat{v}$ given in (24) is the (utility) value with a budget of one dollar.

4 Implications of Saving and Leverage (Intensive Margin)

In this section, we analyze how an investor with a dynamic mental trading account can use savings or leverage to manage his trading strategies over time.

First, we choose the parameter values. A period is a year. As in IJ (2013), we set $\alpha_+ = \alpha_- = 0.5$, $\delta = 5\%$, $\beta = 0.3$, $\mu = 9\%$, $\sigma = 30\%$, and $\theta_s = \theta_p = 1\%$. We set the loss aversion parameter at $\lambda = 1.5$ based on the estimate of Andersen et al. (2022). To target the risk premium $(\mu - r)$ at 6%, a commonly used value for the US stock market and housing risk premia,[20] we set the risk-free rate at $r = 3\%$. We turn off the liquidity shock by setting $\xi = 0$. Finally, we set $\kappa = 0.79$ so that the ratio of debt and liquidating net worth never exceeds 80%.[21] As the stock is the collateral with 20% equity subordination, debt is risk-free. Table 2 summarizes these twelve parameter values.

Table 2: Parameter Values for the Baseline Case. The parameters (r, δ, μ, σ) are continuously compounded and annualized.

α_+	α_-	λ	β	r	δ	μ	σ	θ_s	θ_p	κ	ξ
0.5	0.5	1.5	0.3	3%	5%	9%	30%	1%	1%	0.79	0

[19] In IJ (2013) and HY (2019), the solution features four regions. Voluntary deep-loss realization in their models is impossible. This is because the investor's entire trading account only holds a single stock and selling his stock in deep losses is just too painful.

[20] For example, see Hansen and Singleton (1982) and Mehra and Prescott (1985) for equity risk premium and Piazzesi and Schneider (2016) for housing risk premium.

[21] Since the liquidation value of the stock equals $(1-\theta_s)X$, the investor's net worth upon liquidating the stock is then $W + (1-\theta_s)X$. To target the maximal LTV at 80%, we require the ratio $X/[W + (1-\theta_s)X]$ not to exceed $\phi = 1/(1-0.8) = 5$. Rewriting this leverage constraint yields $W \geq -\kappa X$, where $\kappa = 1 - \theta_s - 1/\phi = 0.79$, as $\phi = 5$ and $\theta_p = 1\%$ for our baseline quantitative analysis.

4.1 Saving in the Risk-free Asset to Spread out Trades over Time

Using parameter values in Table 2, we obtain $w^* = 1.76$, which implies that the investor only allocates $1/(1 + \theta_p + w^*) = 36.1\%$ of his dynamic mental budget to the stock each time he trades and saves the remaining $w^*/(1 + \theta_p + w^*) = 63.5\%$ of his budget for future trades (netting out of the 0.4% trading costs). What determines the level of w^*?

Panel A of Figure 2 shows that $m(w)$ first increases with w for $w \le w^*$, reaches the maximum value $\widehat{v} = 8.53$ at $w^* = 1.76$, and then decreases with w for $w \ge w^*$. The intuition for this single peaked $m(w)$ is as follows. The investor's total (utility) value includes his realization utility from selling the stock and his continuation value. Therefore, choosing a larger w implies a smaller stock allocation $1/(1+\theta_p+w)$ and thus leads to (i) a smaller realization utility and (ii) a slower but also less risky growth of his budget (Π). The continuation value is concave in Π as $\beta = 0.3 < 1$.[22] As w increases, the realization utility flow decreases, but his continuation value increases due to the risk-reduction effect on his future budget Π. This tradeoff pins down w^* (Panel A).

Next, we answer the following counterfactual. How much compensation does an investor require for him to permanently forgo the option to invest in the risk-free asset? We use the fraction Δ of his budget Π_0 to measure this compensation payment.[23]

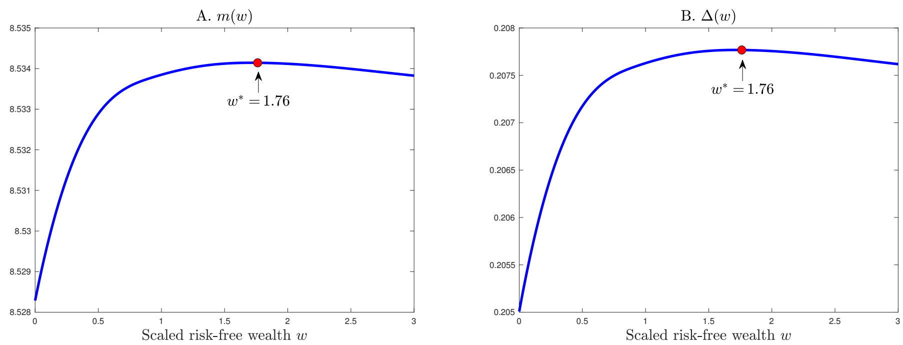

Figure 2: Determining w^* and Quantifying the Value of Saving in the Risk-free Asset. Panels A and B plot $m(w)$ given in (25) and $\Delta(w)$, respectively. Both functions are hump-shaped and maximized at $w^* = 1.76$. See Table 2 for parameter values.

Panel B of Figure 2 shows that $\Delta(\cdot)$ depends on w and indeed has the same monotonicity property as $m(w)$, also peaking at $w^* = 1.76$. Using parameter values in Table 2, we obtain $\Delta(w^*) = 21\%$ at $w^* = 1.76$. That is, the option to save 63.5% of his budget at each trading time and make smaller trades over time is worth about 21% of

[22] Recall that the continuation value is homogeneous of degree β in the investor's budget (Π) at the end of the investment episode.
[23] See Appendix C for details on how to calculate Δ.

the investor's mental budget. While the investor can only hold one stock, the option to spread out his trades over time via saving in the risk-free asset is quite valuable.

Savings Makes the Investor Trade More Frequently and Less Subject to the Disposition Effect. How do savings influence the investor's trading strategies? Panel A of Figure 3 shows that the investor in our model realizes losses much sooner than in IJ (2013): the lower loss-realization boundary $\underline{x}^*$ is 0.69 in our model, much higher than 0.55 in IJ (2013). By saving 63.5% of his budget in the risk-free asset, his downside loss (in dollars) in our model is only about one third of that in IJ (2013), therefore he is more willing to realize losses to reset the reference level B for future gain realizations.[24]

This intuition is corroborated in Panel B of Figure 3, which plots risk attitude measure (RAM), a value function curvature measure defined as:

$$\text{RAM} = -\frac{XV_{XX}}{V_X} = -\frac{xv_{xx}(w^*, x)}{v_x(w^*, x)} = \gamma(x)\,. \tag{27}$$

Because of the homogeneity property, RAM only depends on x, we write RAM as $\gamma(x)$. Panel B of Figure 3 shows that the investor is endogenously risk seeking as his value function is convex and RAM is negative ($\gamma(x) < 0$) in the holding region. Additionally, the investor is more risk seeking in our model than in IJ (2013).

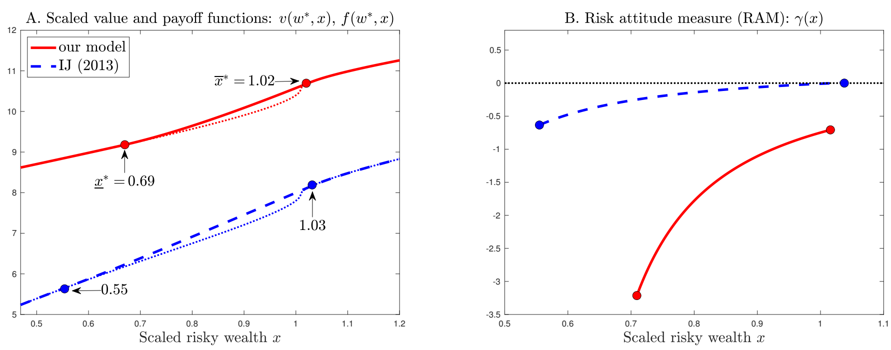

Figure 3: Comparing value functions, payoff functions, and risk attitude measures (RAMs) between our model and the IJ (2013) model. This figure shows that the investor is more willing to realize losses (Panel A) and take risk (Panel B) in our model where he has an option to save in the risk-free asset than in IJ (2013) where he does not. See Table 2 for parameter values.

In sum, the investor with a broader dynamic mental account saves a sizeable fraction of his budget in the risk-free asset and makes smaller stock trades. Additionally, he realizes losses more frequently and is less subject to the disposition effect.

When does the investor want to use leverage rather than to save in the risk-free asset? This is to which we turn next.

[24] The investor in our model realizes gains slightly sooner: The upper boundary $\overline{x}^*$ is 1.02 in our model, which is slightly lower than 1.03 in the IJ (2013) model.

4.2 Using Leverage to Increase Trading Size

In this subsection, we decrease volatility σ from 30% (used in the previous subsection) to 20%, *ceteris paribus*. The optimal allocation in the risk-free asset then becomes $w^* = -0.36 < 0$, which is in sharp contrast to $w^* = 1.76$ when $\sigma = 30\%$. That is, the investor switches from saving 63.5% of his trading budget to borrow 55.4%. With a mental budget of $\Pi_0 = 100$, he invests $X_0 = \Pi_0/(1+\theta_p+w^*) = 154$ (as $w^* = -0.36 < 0$) in the stock he chooses if $\sigma = 20\%$. This trading size is 4.3 times of $X_0 = 36$ when $\sigma = 30\%$ (as $w^* = 1.76$). This exercise shows the very large effect of return volatility σ on the investor's trading strategy. The investor uses leverage when volatility is low because the stock yields a higher Sharpe ratio.

Because leverage significantly increases the investor's exposure to the stock, he becomes more reluctant to realize losses: His loss-realization threshold $\underline{x}^*$ decreases from 0.6 in the IJ (2013) model to 0.47 in our model.[25] The gain-realization threshold $\overline{x}^*$ slightly increases from 1.03 in the IJ (2013) model to 1.04 because of leverage. As a result, the holding region $(\underline{x}^*, \overline{x}^*)$ widens (see Panel A of Figure 4). Consistent with our model's prediction on how leverage impacts gain- and loss-realization strategies, Heimer and Imas (2022) find that access to leverage increases the disposition effect by significantly deferring loss realizations but barely changing gain realizations.

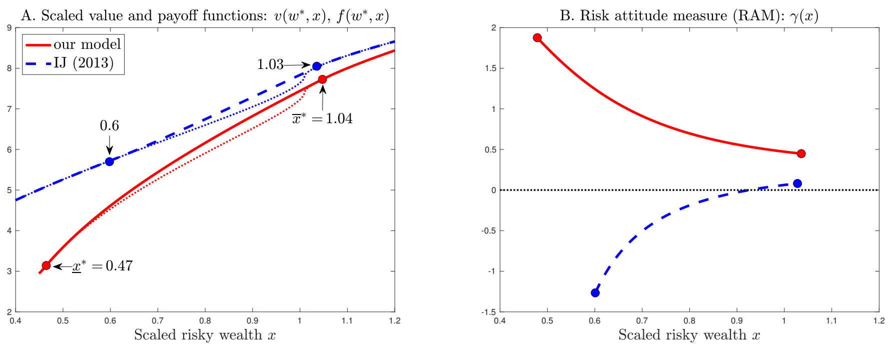

Figure 4: Effect of optimal leverage on value function $v(w^*, x)$ and payoff function $f(w^*, x)$. The optimal w is $w^* = -0.36$. Leverage makes the investor more reluctant to realize losses and more risk averse. The volatility parameter is set at $\sigma = 20\%$. See Table 2 for other parameter values.

Panel B of Figure 4 corroborates our main result about the leverage effect. The investor is risk averse (i.e., $v(w^*, x)$ is concave). Moreover, the RAM $\gamma(x)$ decreases with

[25] Note that the investor still voluntarily realizes losses. The leverage constraint (6) is equivalent to $-\kappa \le W_t/X_t = (W_t/B_t)/(X_t/B_t) = w_t/x_t$ in terms of the scaled variables. So, (6) is not binding if and only if $\underline{x}^* > -w^*/\kappa$. The loss-realization threshold $\underline{x}^* = 0.47$ is slightly larger than the involuntary liquidation threshold: $-w^*/\kappa = 0.36/0.79 \approx 0.46$. That is, the leverage constraint (6) does not bind.

x, which means that the investor becomes less risk-averse as paper losses decrease (x increases). Finally, leverage makes the investor more risk averse. The RAM measure in the IJ (2013) model is lower than in our leverage model for all levels of x. Again, leverage makes the trading size large and the investor endogenously more averse to risk. Finally, we note that quantitatively, the option to use leverage is quite valuable: worth about $\Delta = 31\%$ of the investor's trading budget.

In our preceding analysis, the leverage constraint (6) does not bind; see Footnote 25. This is because $\kappa = 0.79$, which is sufficiently large and the investor's own aversion to losses makes him choose a prudent w^* so it is in his own interest to voluntarily realize losses. What if the leverage constraint is sufficiently tight such that it binds?

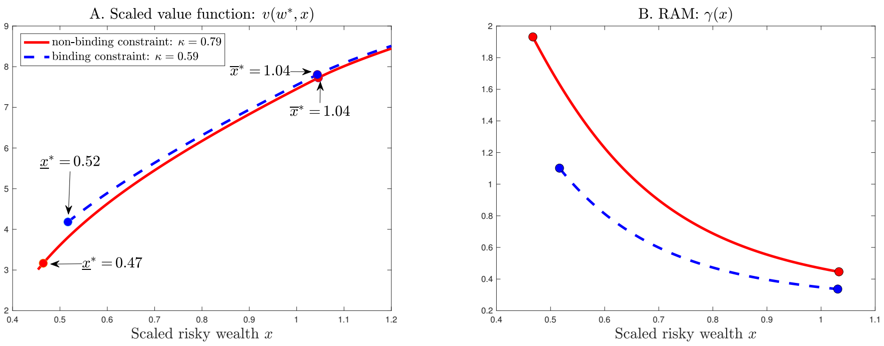

Figure 5: Effect of Leverage Constraints. Leverage constraints can reduce the disposition effect by making the investor realize losses sooner. When $\kappa = 0.79$, the optimal loss realization threshold $\underline{x}^* = 0.47$ and the leverage constraint (6) does not bind. As we tighten the constraint (6) by decreasing κ to 0.59, the optimal loss realization threshold $\underline{x}^*$ increases to 0.52 and the leverage constraint (6) binds. The volatility parameter is set at $\sigma = 20\%$. See Table 2 for other parameter values.

Leverage Constraints. Figure 5 shows that the investor realizes losses sooner when facing a tighter leverage constraint by comparing the $\kappa = 0.79$ and $\kappa = 0.59$ cases.

First, recall that when $\kappa = 0.79$, the investor uses leverage ($w^* = -0.36$) and voluntarily realizes losses when x reaches $\underline{x}^* = 0.47$. In this case, the leverage constraint (6) $x \geq 0.46$ does not bind. Second, as we tighten the constraint (6) by decreasing κ to 0.59, it becomes more likely to bind forcing the investor to realize losses. Anticipating this contingency, the investor prudently reduces his leverage by setting $w^* = -0.31$. The net effect of a tighter leverage constraint (6) and a lower leverage is a higher losses realization threshold: $\underline{x}^* = 0.52$ where the constraint (6) binds. Our model's prediction is consistent with Heimer and Imas (2022) who find that introducing leverage constraints mitigates the disposition effect by making the investor realize losses sooner. Because of a lower leverage ratio, the investor is less averse to realize losses as we can see by comparing the two lines for RAMs in Panel B of Figure 5.

In contrast, the gain-realization threshold $\overline{x}^*$ remains at 1.04. This is consistent with our prior result that the investor's gain realization strategy is not sensitive to changes in his trading opportunity. This is because the investor's preference is concave in gains (in the gain region), the benefit of locking in gains is thus sufficiently large.

In sum, our model predicts that (i) investors using leverage have stronger disposition effects and (ii) introducing leverage constraints can reduce investors' disposition effects by disciplining their behavioral biases. These predictions are consistent with the findings in Barber et al. (2019) and Heimer and Imas (2022).

We have thus far analyzed how the investor's dynamic mental account allows him to use savings and leverage to manage his trading strategies by separating his trading size from his trading budget. Next, we study how the investor's saving/leverage and trading strategies vary as we change the stock return volatility σ. We show that the *interaction* between stock return volatility and the investor's endogenous response via dynamically managing his broader mental account generates new predictions on the investor's trading and risk management policies.

4.3 Does the Investor Prefer More or Less Volatile Stocks?

To answer the question posed in this subsection's title, we calculate the (utility) value $\widehat{v}$ defined in (24)-(25) for a range of σ and plot this relation in Panel A of Figure 6.

We see that the (scaled) value $\widehat{v}$ at the moment of trading first decreases with σ, reaches the minimal value of 7.95 when $\sigma = 25\%$, and then increases with σ for $\sigma \geq 25\%$ (the red solid line in Panel A). The minimal value of $\widehat{v} = 7.95$ is attained in our model at the solid black dot connecting the two lines indicating that our model solution agrees with the solution in IJ (2013) where the investor has no option to save nor to use leverage when $\sigma = 25\%$. This is because the investor optimally chooses $w^* = 0$ when $\sigma = 25\%$. In this case, it is optimal for the investor to neither save or use leverage. The optimal dynamic mental account management calls for an all-in stock strategy.

For $\sigma \leq 25\%$, the investor in our model uses leverage to increase his (dollar) exposure to the stock (note that $w^* < 0$ to the left of the solid black dot in Panel B of Figure 6.) Because of his larger risk exposure, the investor is endogenously more risk averse as he may be forced to realize losses. This is why he prefers less risky stocks after using leverage, explaining why his value function $\widehat{v}$ *decreases* with σ in this range.

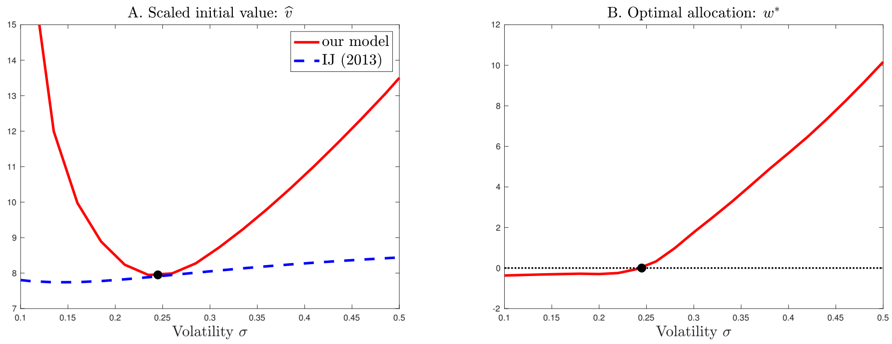

Figure 6: SCALED VALUE $\widehat{v}$ AND OPTIMAL w^* AS FUNCTIONS OF VOLATILITY σ. Panel A shows that $\widehat{v}$ decreases with σ when $\sigma < 25\%$ but increases with σ when $\sigma > 25\%$. This is because the investor (1.) uses leverage when $\sigma < 25\%$ and the value of leverage decreases with σ and (2.) saves in the risk-free asset when $\sigma > 25\%$ and the value of saving increases with σ (Panel B). Finally, the value of saving and leverage is zero when $\sigma = 25\%$. See Table 2 for parameter values other than σ.

In contrast, for $\sigma \geq 25\%$, the investor saves a fraction of his budget for his future trading opportunities (note that $w^* > 0$ to the right of the solid black dot in Panel B), thus decreasing his risk exposure. Because of his reduced exposure to the stock, the investor is more willing to take on a more risky stock, explaining why $\widehat{v}$ increases with σ in the high-σ range.

An important takeaway from Figure 6 is that investors who use leverage prefer stocks with low volatility, while investors who save to spread his trades over time prefer riskier stocks. This prediction is consistent with Bian et al. (2021) who find that stocks bought in margin accounts tend to have lower systematic volatility and total volatility than stocks bought in cash accounts. The balancing act between risk taking at the trading-account level (the size of w^*) and at the stock level (higher or lower σ) yields an optimal risk exposure to the investor. This balancing reflects the interaction between the two layers of mental accounts in our model: When an investor has both a stock-level mental account for each utility-burst calculation and a broader intertemporal mental budget for his trading account, he uses savings or leverage to target a volatility level for his trading account. The endogenous response of the investor's risk taking given the volatility of stocks that he trades is at the core of our model's mechanism.

5 A Jump-Diffusion Model

In this section, we analyze the effects of stock-price jumps on investors' trading.[26] We highlight two key predictions. The first prediction is that the investor voluntarily sells a

[26] We turn off liquidity shocks by setting $\xi = 0$.

stock in deep losses, when he has sufficiently high savings in his trading account. Even when the stock he owns is in a deep loss, his trading account is only in moderate or even small losses. What happens if the investor's savings are not sufficiently high?

In this case, the investor is unwilling to sell a stock in deep losses. However, he may be willing to sell the stock after its price rebounds a bit. This is because the value of his broader mental account, the sum of his savings and the value of his stockholdings, increases just enough with the rebound, which makes him willing to realize losses for future gain realizations. This sell-after-rebound is the second prediction of our model.

In sum, these two predictions, generated by two cases of our model solution, arise from the interaction between the two-layered mental-accounting structure of our model.

Jump-Diffusion Model. We model the price process for stock n, where $n \in \{1, 2, \cdots, N\}$ by incorporating jumps into the GBM process given in (1) as follows:

$$\frac{dP_{n,t}}{P_{n,t-}} = \mu dt + \sigma d\mathcal{Z}_{n,t} - (1-Y)d\mathcal{J}_{n,t}\,, \quad P_0 > 0\,, \tag{28}$$

where $\mathcal{J}_n$ is a pure jump with an arrival rate (ρ) and the random variable $Y \in [0, 1]$ is drawn from a cumulative distribution function (cdf), $\Omega(Y)$. Let $\tau^{\mathcal{J}}$ denote the jump arrival time. If a jump occurs at t ($d\mathcal{J}_{n,t} = 1$), the stock price falls from $P_{n,t-}$ to $P_{n,t} = YP_{n,t-}$. Other assumptions are the same as in our diffusion model (Section 2).[27]

Solution. As in our diffusion model, using the homogeneity property, we work with the scaled state variables ($w = W/B$ and $x = X/B$) and scaled value functions: $v(w, x)$ and $f(w, x)$. While $dw_t = 0$, a jump term appears in the x_t process:

$$dx_t/x_{t-} = (\mu - r)dt + \sigma d\mathcal{Z}_{n,t} - (1-Y)d\mathcal{J}_{n,t}\,. \tag{29}$$

The solution has two domains. In the holding domain $\mathcal{H}$ where $v(w, x) > f(w, x)$, the investor holds onto his stock position and $v(w, x)$ solves the following HJB equation:

$$\delta_e v(w,x) = \frac{\sigma^2 x^2}{2} v_{xx}(w,x) + (\mu - r)x v_x(w,x) + \rho\left(\mathbb{E}[v(w,Yx)] - v(w,x)\right), \tag{30}$$

where $\delta_e = \delta - \beta r$ is the investor's effective discount rate. In the realization domain $\mathcal{R}$, $v(w, x) = f(w, x)$, where the scaled payoff function $f(w, x)$ is given by (23).

Solution wise, there are two cases: (a) the three-region case and (b) the four-region case. Both cases feature a gain-realization region, a normal holding region, and a loss-realization region.[28] The key difference between the two cases boils down to whether it is

[27] To ease exposition, we focus on the case where the investor chooses to save rather than use leverage.

[28] It is possible that this loss-realization region $\mathcal{R}_-$ does not exist, e.g., when $\alpha_+ = \alpha_- = \beta = 1$, as in BX (2012). We consider this two-region case as a special subcase of the four-region case.

optimal for the investor to voluntarily realize deep stock losses when x is close to zero. If yes, the solution features three regions. Otherwise, there is a fourth region: the deep-loss holding region. Before analyzing the two cases, we first choose parameter values.

Parameter Choices. We specify the cdf $\Omega(Y)$ for $Y \in [0,1]$ using a widely-used power law as in Barro (2006) and the rare-disaster literature: $\Omega(Y) = Y^{\psi}$, where $\psi > 0$ is the power-law parameter. We set ψ at 6.3 as in Barro and Jin (2011), which implies an expected 14% decrease of stock prices for each jump: $\mathbb{E}(1-Y) = \frac{1}{\psi+1} = 14\%$. We set the jump arrival rate $\rho = 0.73$ per annum, to capture one jump arrival for about 1.4 years on average (Huang and Huang, 2012). We set $\lambda = 2.25$ for loss aversion (Tversky and Kahneman, 1992) and target a risk premium of 6%, which yields $\mu = 19\%$.[29] To illustrate very different economic predictions for the two cases, we consider two levels of volatility: $\sigma = 30\%$ and $\sigma = 24\%$. All other parameter values are given in Table 2.

5.1 Case a: Three-region Solution ($\sigma = 30\%$)

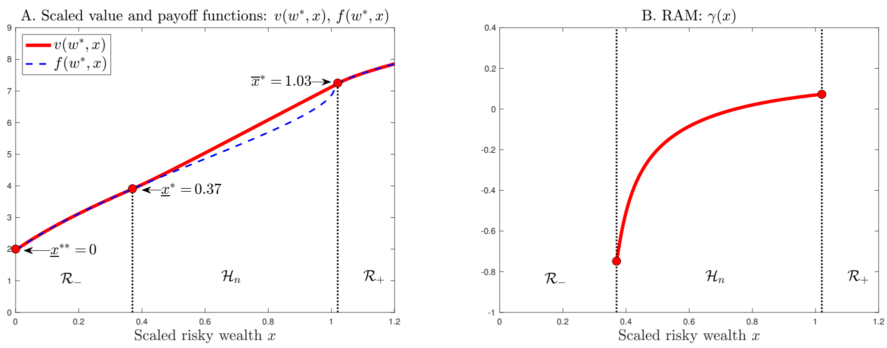

Figure 7: THREE-REGION SOLUTION FOR A JUMP-DIFFUSION MODEL. Unlike diffusion models, the investor may voluntarily realize *deep* losses (where x is close to zero) on the optimal path. The investor saves 20.5% of his budget in the risk-free asset: $w^* = 0.26$. Realizing deep losses resets the reference level for future gains realizations, which outweighs high utility costs from realizing deep losses. Panel A plots $v(w^*,x)$ and $f(w^*,x)$. Panel B plots the RAM: $\gamma(x)$.

When $\sigma = 30\%$, the solution features three regions. First, the investor realizes gains when x_t exceeds the gain-realization threshold $\overline{x}^* = 1.03$, similar to our diffusion models. Second, the investor holds onto his stock position in the $x \in (0.37, 1.03)$ region. This holding region is wide because of the investor's strong aversion to realize losses. These two regions are standard as in diffusion models (see, e.g., in IJ (2013) and Section 2).

In the third region where $x \in (0, 0.37)$, the investor voluntarily realizes losses. Importantly, even when stock losses are close to 100%, realizing losses is optimal. This is

[29] This follows from $r = 3\%$ and $\mu - \rho\,(1 - \mathbb{E}[Y]) - r = 6\%$.

because 20.5% of the investor's budget is allocated to the risk-free asset when he trades ($w^* = 0.26$). With 20.5% set aside in savings, the value of resetting the reference level for future gain realizations is then larger than the utility costs of realizing deep losses. Specifically, the value function at $x = 0$ is positive: $v(w^*, 0) = f(w^*, 0) = 2 > 0$.

Selling a Stock in Deep Losses. Voluntarily selling a stock in deep losses is a unique prediction of our model. To generate voluntary deep-loss realizations, it is necessary for the model to have both (downward) asset-price jumps and an option for the investor to save in the risk-free asset. The intuition is as follows. First, without jumps, the deep-loss region cannot be reached on the optimal path. Second, without savings in his trading account, it is always optimal for the investor to hold onto his stock positions in the deep-loss region as realizing losses is so painful and there is almost no future. In contrast, with savings in his trading account, a deep loss at an individual stock level does not imply a deep loss in his trading account. An investor with a dynamically evolving mental budget for his trading account is thus willing to sell a stock in deep losses for future gain realizations at the trading account level. This prediction is broadly consistent with An et al. (forthcoming) who find that the disposition effect is large when the portfolio is at a loss but nearly disappears when the portfolio is at a gain.[30]

As in IJ (2013), our model also predicts that the probability of selling a stock increases as its paper gains or losses increase, generating a V-shaped selling propensity pattern as shown by Ben-David and Hirshleifer (2012) and An (2016).[31]

Next, we analyze case b which features four regions. Although models in the literature also feature four regions, the time-series predictions of our case b are quite different due to jumps and the interaction between the two layers of the mental accounts.

5.2 Case b: Four-region Solution ($\sigma = 24\%$)

By decreasing σ to 24%, we obtain a four-region solution (Figure 8), one more region than for case a where $\sigma = 30\%$ (Figure 7). The gain-realization region where $x \geq 1.03$ and the normal holding region where $x \in (0.34, 1.03)$ are qualitatively similar in cases a and b. The other parts of the solution for the two cases are quite different.

[30] Hartzmark (2015) documents the rank effect for the disposition effect: individuals are more likely to sell the extreme winning and extreme losing positions in their portfolio.

[31] As in IJ (2013), to generate this V-shaped selling propensity, we need heterogeneous preferences. Liu et al. (2022) provide survey results in support of preference heterogeneity.

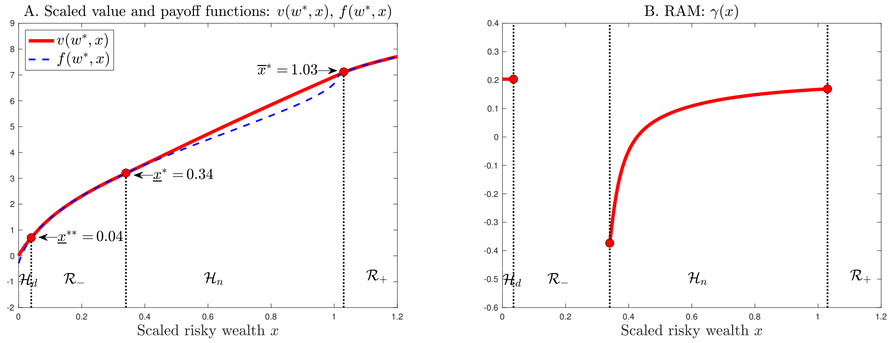

Figure 8: FOUR-REGION SOLUTION FOR A JUMP-DIFFUSION MODEL. The investor saves a tiny fraction (1.9%) of his budget in the risk-free asset: $w^* = 0.02$. In addition to the gain-realization and loss-realization regions as well as the normal holding region that lies between these two realization regions, there is a fourth deep-loss holding region. Unlike diffusion models, the investor may find himself in the fourth deep-loss holding region on the optimal path. Other than $\sigma = 24\%$, all other parameters are the same as for Figure 8. Panel A plots $v(w^*, x)$ and $f(w^*, x)$. Panel B plots the RAM: $\gamma(x)$.

To the left of $\underline{x}^* = 0.34$ is the loss-realization region: $x \in (0.04, 0.34)$, where the investor voluntarily realizes losses. Finally, in the far left region where $x \in (0, 0.04)$, the stock is in deep losses and the investor passively holds onto his stock. In this region, realizing losses is too painful, as he only has 1.9% of his budget in savings ($w^* = 0.02$).[32] This holding prediction in the deep-loss region (where $x \in (0, 0.04)$) is the opposite of the stock-selling prediction of case a in the same region. Finally, the investor only realizes losses in case b when $x \in (0.04, 0.34)$, suggesting that his loss-realization propensity is non-monotonic in x. Next, we discuss the new time-series prediction of case b.

Selling a Stock after It Rebounds. Consider an investor whose stock in his trading account is in a very deep loss: $x = 0.03$. While holding the stock as $x < \underline{x}^{**} = 0.04$ (see Figure 8), the stock rebounds, cutting his losses a bit and bringing his x up to 0.04, the investor then immediately realizes his stock losses as he has entered the loss-realization region: $x \in [0.04, 0.34]$. Realizing this loss allows him to reset his reference level and start anew from $x = 1$ in the normal holding region. This process repeats.

This path captures the following prediction that would not have been possible in diffusion models. The investor, while unwilling to sell his stock in deep losses, voluntarily sells the stock after it rebounds a bit. Intuitively, a rebound cuts his losses by just enough and is all he needs to enter into the loss-realization region. Although direct empirical evidence is not known yet, this testable prediction is consistent with our observation that retail investors often sell their losing stocks after these stocks rebound a little. Finally, we compare our case b with diffusion models.

[32] Since it is optimal to hold the stock in this region, $v(w^*, 0) = 0$ and $f(w^*, 0) < 0$ (see Figure 8).

Comparing Case b (Featuring Four-region Solution) with Diffusion Models. Recall that solutions of diffusion models with S-shaped realization utilities, e.g., IJ (2013), HY (2019), and our model in Section 2, also feature four regions. What are then the differences between our jump-diffusion model and these diffusion models?

In diffusion models, the deep-loss holding region and the loss-realization region (other than the upper loss-realization threshold of this region) are never reached on the optimal path. This is because diffusion processes are continuous and the x_t process never falls below the upper (right) boundary $\underline{x}^*$ for the loss-realization region due to optimality. In contrast, in our jump-diffusion model, because x_t in our model can jump downward at any time t, all four regions are on the optimal path. Indeed any value of x lower than the gain-realization threshold $\overline{x}^*$ can be on the optimal path in our jump-diffusion model.

Comparing Case a with Case b. Finally, we conclude this section by highlighting a key difference between the two cases, which is whether the investor is willing to sell his stock in deep losses. The key is whether the investor has set aside sufficient savings in the investor's dynamic trading account. Recall that $w^* = 0.26$ in case a where $\sigma = 30\%$ (Figure 7) and $w^* = 0.02$ in case b where $\sigma = 24\%$ (Figure 8). As stock volatility σ decreases, the demand for savings (w^*) decreases, which means the incentive to realize losses so as to reset the reference level for future gain realizations decreases. In sum, in our model with two layers of mental accounts, the investor's demand for savings plays a crucial role in generating new time-series predictions.

6 Piecewise Linear Realization Utility

In this section, we analyze the case with piecewise linear realization utility.[33] We obtain the following two main results. First, the investor uses leverage to increase his exposure to the stock he chooses. Second, he realizes losses only when he is forced by a binding leverage constraint (6). That is, the result that an investor with piecewise linear utility never voluntarily realizes losses in BX (2012) continues to hold in our model. The following proposition summarizes these two results.

Proposition 1. *In the absence of liquidity shocks, for piecewise linear realization utility $U(G, B)$ ($\alpha_{\pm} = \beta = 1$), the investor uses leverage and never saves: $w^* \leq 0$. Additionally, he does not realize losses until the leverage constraint (6) binds, i.e., when $x = -w^*/\kappa$.*

[33] To ease exposition, we turn off liquidity shocks by setting $\xi = 0$.

To ease comparison, we keep the parameter values the same as in our baseline model whenever feasible. We set $\delta = 35\%$ in order to satisfy the transversality condition (see Internet Appendix IC). We show that the investor highly values leverage and adopts very different gain/loss realization policies compared with the BX (2012) model.

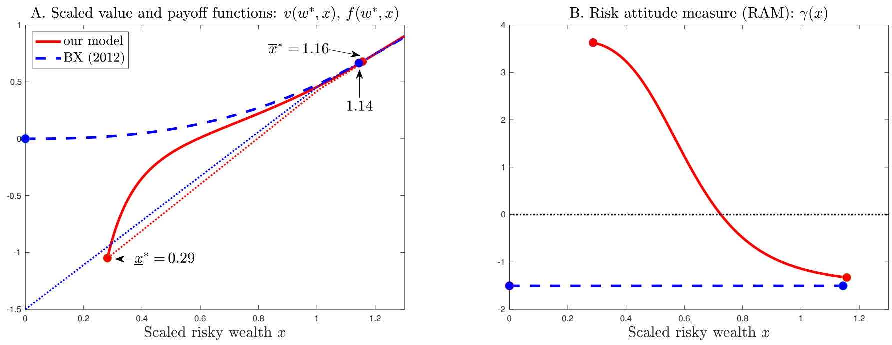

Figure 9: Piecewise Linear Realization Utility: $\alpha_\pm = \beta = 1$. The optimal w is $w^* = -0.23$ and the investor realizes losses only when the leverage constraint (6) binds. The discount rate is set at $\delta = 35\%$ for convergence. For other parameter values, see Table 2.

Figure 9 compares our model solution with the BX (2012) model solution. First, recall that the value function in BX (2012) is convex and thus the RAM is negative: $\gamma(x) = -1.51$ (the dashed blue lines in Panels A and B).[34] This is because realizing losses is too painful compared with the benefit of resetting the reference level B for gain realizations in the future. Since the investor only realizes gains and never realizes losses, he prefers higher volatility σ, *ceteris paribus.*

As the investor is risk seeking in BX (2012) and can borrow in our model, it is optimal for him to use some leverage. However, high leverage makes his holdings volatile, which increases the probability of forced loss realization (as the leverage constraint is more likely to bind sooner). Because of the large utility cost of forced loss realization, the investor chooses a prudent leverage ratio by setting $w^* = -0.23$ and thus only realizes losses if x falls by 71% to $-w^*/\kappa = 0.23/0.79 = 29\%$ (the solid red line in Panel A).[35]

In sum, the investor is risk averse ($\gamma(x) > 0$) in the $x \in (0.29, 0.73)$ region where he is close to the leverage constraint and the incentive to manage downside risk is strong. However, he is risk loving ($\gamma(x) > 0$) in the $x \in (0.73, 1.16)$ region where he is sufficiently far away from the constraint and the risk-taking force in BX (2012) dominates. The solid red lines in the two panels of Figure 9 depict this highly non-monotonic risk-taking

[34] Note that the investor's value equals zero at the origin. This is because $x = 0$ is an absorbing state and no loss realization is optimal.

[35] An investor with an initial budget of $\Pi_0 = 100$ allocates $X_0 = \Pi_0/(1 + \theta_p + w^*) = 128$ to the stock and finances $28/128 = 22\%$ of his stock holdings with leverage.

incentive. Note that the option to use leverage turns a convex value function in BX (2012) to a concave-then-convex shaped value function. Finally, the value of using leverage is quantitatively substantial: worth $\Delta = 23\%$ of the investor's budget.

7 Conclusion

Building on Barberis and Xiong (2009, 2012) and Ingersoll and Jin (2013), we develop a jump-diffusion model where an investor receives utility bursts from realizing stock gains and losses. In addition to a series of stock-level mental accounts for each utility burst, the investor also has a mental budget that brackets all his investment episodes together to evaluate his intertemporal realization utility. A key implication of our two-layer mental-account model is that the investor does not have to use his entire mental budget when trading stocks. Instead, he can save a fraction of his trading budget and/or use leverage to separate his trading size from his trading budget.

By saving a fraction of his budget ($w^* > 0$), the investor makes smaller trades and spreads out his trades over time, lowers transaction costs (in dollars), and is less subject to the disposition effect. By using leverage ($w^* < 0$), the investor increases his trading size beyond his budget. Because of increased risk exposures and larger transaction costs, the investor is more reluctant to realize losses, strengthening the disposition effect (Barber et al., 2019; Heimer and Imas, 2022). Introducing leverage constraints mitigates it by making the investor realize losses sooner (Heimer and Imas, 2022).

Our model generates new predictions for an investor holding a stock in deep losses. With enough savings in his mental trading account, he voluntarily sells the stock in deep losses. This prediction is consistent with the portfolio-driven disposition effect (An et al., forthcoming). If his savings in his trading account are low, the investor is unwilling to sell the stock in deep losses, but realizes dampened losses after the stock he owns rebounds a bit. This sell-after-rebound prediction is consistent with our observation that retail investors often sell their losing stocks after these stocks rebound a bit.

Quantitatively, we find that having access to the risk-free asset is worth over 20% of the investor's total wealth in our calibrated diffusion models. Incorporating jumps further significantly enhances the quantitative importance of our model mechanism.

In our model, the investor holds or trades a single stock at each moment. In reality, investors hold and trade multiple stocks. We plan to study how a realization-utility investor trades in this richer setting.

References

Altarovici, Albert, Max Reppen, and H. Mete Soner. 2017. Optimal consumption and investment with fixed and proportional transaction costs. *SIAM Journal on Control and Optimization* 55 (3):1673–1710.

An, Li. 2016. Asset pricing when traders sell extreme winners and losers. *Review of Finacial Studies* 29 (3):823–861.

An, Li, Joseph Engelberg, Matthew Henriksson, Baolian Wang, and Jared Williams. forthcoming. The portfolio-driven disposition effect. *Journal of Finance* .

Andersen, Steffen, Cristian Badarinza, Lu Liu, Julie Marx, and Tarun Ramadorai. 2022. Reference dependence in the housing market. *American Economic Review* 112 (10):3398–3440.

Barber, Brad M., Xing Huang, Jeremy Ko, and Terrance Odean. 2019. Leveraging overconfidence. *Working Paper* .

Barberis, Nicholas and Wei Xiong. 2009. What drives the disposition effect? An analysis of a long-standing preference-based explanation. *Journal of Finance* 64 (2):751–784.

———. 2012. Realization utility. *Journal of Financial Economics* 104:251–271.

Barro, Robert J. 2006. Rare disasters and asset markets in the twentieth century. *Quarterly Journal of Economics* 121 (3):823–866.

Barro, Robert J. and Tao Jin. 2011. On the size distribution of macroeconomic disasters. *Econometrica* 79 (5):1567–1589.

Ben-David, Itzhak and David Hirshleifer. 2012. Are investors really reluctant to realize their losses? Trading responses to past returns and the disposition effect. *Review of Financial Studies* 25 (8):2485–2532.

Bian, Jiangze, Zhi Da, Zhiguo He, Dong Lou, Kelly Shue, and Hao Zhou. 2021. Margin trading and leverage management. *Working Paper* .

Crandall, Michael, Hitoshi Ishii, and Pierre-Louis Lions. 1992. User's guide to vicosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* 27:1–67.

Dai, Min and Yifei Zhong. 2010. Penalty methods for continuous-time portfolio selection with proportional transaction costs. *Journal of Computational Finance* 13 (3):1–31.

Dixit, Avinash K. and Robert S. Pindyck. 1994. *Investment under Uncertainty.* Princeton University Press.

Evans, Lawrence C. 2010. *Partial differential equations: Second edition.* American Mathematical Society.

Genesove, David and Christopher Mayer. 2001. Loss aversion and seller behavior: Evidence from the housing market. *Quarterly Journal of Economics* 116 (4):1233–1260.

Hansen, Lars Peter and Kenneth J. Singleton. 1982. Generalized instrumental variables estimation of nonlinear rational expectations models. *Econometrica* 50 (5):1269–1286.

Hartzmark, Samuel M. 2015. The worst, the best, ignoring all the rest: The rank effect and trading behavior. *Review of Financial Studies* 28 (4):1024–1059.

He, Xuedong and Linan Yang. 2019. Realization utility with adaptive reference points. *Mathematical Finance* 29:409–447.

Heath, Chip, Steven Huddart, and Mark Lang. 1999. Psychological factors and stock option exercise. *Quarterly Journal of Economics* 114 (2):601–627.

Heimer, Rawley Z. and Alex Imas. 2022. Biased by choice: How financial constraints can reduce financial mistakes. *Review of Financial Studies* 35 (4):1643–1681.

Huang, Jing-Zhi and Ming Huang. 2012. How much of the corporate-treasury yield spread is due to credit risk? *Review of Asset Pricing Studies* 2 (2):153–202.

Ingersoll, Jonathan E. and Lawrence J. Jin. 2013. Realization utility with reference-dependent preferences. *Review of Financial Studies* 26 (3):723–767.

Kahneman, Daniel and Dan Lovallo. 1993. Timid choices and bold forecasts: A cognitive perspective on risk taking. *Management Science* 39 (1):17–31.

Kahneman, Daniel and Amos Tversky. 1979. Prospect theory: An analysis of decision under risk. *Econometrica* 47 (2):791–812.

Kyle, Albert S., Hui Ou-Yang, and Wei Xiong. 2006. Prospect theory and liquidation decisions. *Journal of Economic Theory* 129:273–288.

Li, Yan and Liyan Yang. 2013. Prospect theory, the disposition effect, and asset prices. *Journal of Financial Economics* 107:715–739.

Liu, Hongqi, Cameron Peng, Wei A. Xiong, and Wei Xiong. 2022. Taming the bias zoo. *Journal of Financial Economics* 143:716–741.

McDonald, Robert and Daniel Siegel. 1986. The value of waiting to invest. *Quarterly Journal of Economics* 101 (4):707–728.

Mehra, Rajnish and Edward C. Prescott. 1985. The equity premium: A puzzle. *Journal of Monetary Economics* 2:145–161.

Odean, Terrance. 1998. Are investors reluctant to realize their losses? *Journal of Finance* 53 (5):1775–1798.

Øksendal, Bernt and Agnes Sulem. 2002. Optimal consumption and portfolio with both fixed and proportional transaction costs. *SIAM Journal on Control and Optimization* 40 (6):1765–1790.

Pham, Huyên. 2009. *Continuous-time Stochastic Control and Optimization with Financial Applications*. Springer.

Piazzesi, Monika and Martin Schneider. 2016. Housing and macroeconomics. *Handbook of Macroeconomics* 2:1547–1640.

Rabin, Matthew and Dimitri Vayanos. 2010. The Gambler's and Hot-Hand Fallacies: Theory and Applications. *Review of Economic Studies* 77:730–778.

Read, Daniel, George Loewenstein, and Matthew Rabin. 1999. Choice Bracketing. *Journal of Risk and Uncertainty* 19 (1-3):171–197.

Shefrin, Hersh and Meir Statman. 1985. The disposition to sell winners too early and ride losers too long: Theory and evidence. *Journal of Finance* 40 (3):777–790.

Thaler, Richard H. 1999. Mental accounting matters. *Journal of Behavioral Decision Making* 12 (3):183–206.

Thaler, Richard H. and Eric J. Johnson. 1990. Gambling with the house money and trying to break even: The effects of prior outcomes on risky choice. *Management Science* 36 (6):463–660.

Tversky, Amos and Daniel Kahneman. 1992. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty* 5:297–323.

Appendices

A Variational Inequality

We first use a variational inequality to characterize the value function $V(W, X, B)$ defined by (14) or equivalently (17) and then discuss transversality conditions. For proofs of existence and uniqueness of the variational inequality as well as the associated verification theorem, we refer readers to Internet Appendix ID.[36]

A.1 Characterization by Variational Inequality

In the region where $W > -\kappa X$, $X \geq 0$, and $B > 0$, the leverage constraint (6) does not bind and the value function $V(W, X, B)$ defined in (14) satisfies the following variational inequality (see, e.g., Pham (2009) on the standard optimal-stopping theory):

$$\max\left\{\mathcal{L}V(W, X, B),\ F(W, X, B) - V(W, X, B)\right\} = 0 \tag{A.1}$$

where

$$\mathcal{L}V = \frac{1}{2}\sigma^2 X^2 V_{XX} + \mu X V_X + rWV_W + rBV_B - \delta V + \xi\left[U(G,\, B) - V\right] \tag{A.2}$$

and $G = (1 - \theta_s)X - B$ is the realized gain (if positive) or loss (if negative).

The intuition for (A.1) is as follows. At each time t, the investor can either keep the holdings in his trading account unchanged or sell the stock he owns to realize a gain or loss. If it is optimal to keep his holdings unchanged, the first term in (A.1) is larger than the second term, which implies the standard HJB equation in the waiting/holding region, $\mathcal{L}V(W, X, B) = 0$, holds. On the other hand, if it is optimal to sell the stock to realize a gain or loss, we must have $V(W, X, B) = F(W, X, B)$. This is the case where the second term in (A.1) is larger than the first term. As either keeping holdings unchanged or trading must be optimal at any given t, the variational inequality (A.1) holds at all time.

Finally, when the leverage constraint (6) binds, the investor has no choice but to realize losses in order to satisfy the leverage constraint. Therefore,

$$V(W, X, B) = F(W, X, B)\ , \quad \text{when}\ \ X = -W/\kappa > 0\,. \tag{A.3}$$

[36] The optimization problem (14) is an impulse-control problem (Øksendal and Sulem, 2002; Altarovici et al., 2017) and the solution to the variational inequality (A.1) should be interpreted in a weak sense, i.e., viscosity solutions (Crandall et al., 1992).

Homogeneity. We can simplify the optimization problem as follows. First, in the domain $\mathcal{S} = \{x > 0,\ w > -\kappa x\}$ where the leverage constraint (6) does not bind, the variational inequality (A.1) for $V(W, X, B)$ is equivalent to the following simplified variational inequality for the scaled value function $v(w, x) = B^{-\beta} V(W, X, B)$:

$$\max\{\mathscr{L}v(w,x),\ f(w,x) - v(w,x)\} = 0\,, \tag{A.4}$$

where $f(w, x)$ is the scaled payoff function given in (23), $\mathscr{L}v$ is given by

$$\mathscr{L}v = \frac{1}{2}\sigma^2 x^2 v_{xx} + (\mu - r)x v_x - \delta_e v + \xi\,[u((1-\theta_s)x - 1) - v]\,, \tag{A.5}$$

and $\delta_e = \delta - \beta r$. Finally, if the leverage constraint (6) binds, i.e., when $x = -w/\kappa > 0$, (A.3) is simplified to $v(w, x) = f(w, x)$.

A.2 Transversality Conditions

Next, we provide transversality conditions which ensure that the value functions are finite. For brevity, we only report results for our diffusion model with liquidity shocks (in Section 2). We provide transversality conditions for our jump-diffusion model in Internet Appendix IC.

It is convenient to introduce the following notation:

$$K = \frac{1-\theta_s}{1-\theta_s-\kappa}\ . \tag{A.6}$$

We propose the following transversality conditions for our diffusion model.

(i) If $\beta = 1$,

$$\delta + \xi > \mu + \max\{0, (\mu - r)(K-1)\}\ , \tag{A.7}$$

where K is defined in (A.6).

(ii) If $\beta < 1$ and $\frac{\mu - r}{(1-\beta)\sigma^2} \in (0, K)$,

$$\delta + \xi > \beta r + \max\left\{0,\ \frac{\sigma^2}{2}\alpha_+(\alpha_+ - 1) + (\mu - r)\alpha_+,\ \frac{\beta(\mu-r)^2}{2(1-\beta)\sigma^2}\right\}. \tag{A.8}$$

(iii) If $\beta < 1$ and $\frac{\mu - r}{(1-\beta)\sigma^2} \notin (0, K)$,

$$\delta + \xi > \beta r + \max\left\{0, \frac{\sigma^2}{2}\alpha_+(\alpha_+ - 1) + (\mu - r)\alpha_+, \frac{\sigma^2}{2}\beta(\beta-1)K^2 + (\mu - r)\beta K\right\}. \tag{A.9}$$

Next, we use a two-step procedure to verify these transversality conditions. First, we construct a sufficiently smooth supersolution of the variational inequality (A.4) satisfying (A.7)-(A.9). Second, we prove that the value function is bounded from above by the supersolution. Internet Appendix IC provides calculation details.

Next, we turn to the voluntary participation condition introduced in Section 2. Specifically, we characterize the set of drift and volatility parameters (μ, σ) in which the investor is willing to invest a fraction of his mental budget in stocks and the remaining in the risk-free asset. (Leverage is possible provided that the leverage constraint is satisfied.)

A.3 Participation Condition: Drift and Volatility Effects

Recall that the participation condition is $\widehat{v} > 0$. In Figure A.1, we characterize the participation condition $\widehat{v} > 0$ by focusing on the drift μ and volatility σ parameters (see Table 2 for all other parameter values). The solid red line divides the admissible domain for (μ, σ) into two parts. Above the line is the set of (μ, σ) pairs where the investor voluntarily invests in stocks. Below this line he does not invest in stocks implying that his realization utility equals zero. When investing in a stock he chooses, he allocates a fraction of his budget to the stock and saves his remaining (mental) trading budget. It is possible that he allocates more than 100% of his mental budget to the stock he chooses, using leverage subject to the leverage constraint (6).

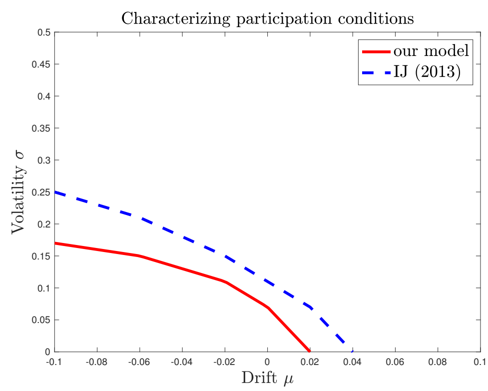

Figure A.1: Voluntary Participation Conditions for Diffusion Models: $\widehat{v} > 0$. The solid red line divides the (μ, σ) plane for our model. Above this solid line, the investor voluntarily holds a combination of the risk-free asset and the stock he chooses so that $\widehat{v} > 0$ is satisfied. The dashed blue line divides the (μ, σ) plane for the corresponding realization-utility models where the investor at each t has a binary choice of either holding only the stock or investing solely in the riskfree asset. Above this dashed line, the participation condition $\widehat{v} > 0$ is satisfied. Parameter values are: $\delta = 0.05$, $\xi = 0.1$, $\kappa = 0.79$, $r = 3\%$, $\theta_p = \theta_s = 1\%$, $\alpha_+ = 0.5, \alpha_- = 0.5, \lambda = 1.5$, and $\beta = 0.3$.

To highlight how our model relaxes the investor's voluntary participation constraint $\widehat{v} > 0$ (increasing the set of admissible parameter values), we also characterize the participation condition for realization-utility models, e.g., BX (2012), IJ (2013), and HY (2019), where the investor has a *binary* choice between two options: allocating his entire budget either to a stock he chooses or to the risk-free asset (a combination of the two assets is not allowed). The participation condition in this case is satisfied if and only if the (μ, σ) parameters are in the region above the dashed blue line.

The solid red line which defines the participation condition for our model is lower than the dashed blue line for realization-utility models in the literature where the investor has a *binary* choice between two options. This result is consistent with our intuition that the participation condition $\widehat{v} > 0$ for the investor in our model is easier to satisfy because of the additional investment flexibility that he has: the intensive margin (how much to invest in a stock that he chooses).

B Solutions and Transition Dynamics

In this appendix, we first derive closed-from solutions for our baseline model with no liquidity shocks and then analyze the model-implied transition dynamics. Finally, for the generalized jump-diffusion model, we propose a numerical procedure based on the penalty method proposed in Dai and Zhong (2010).

B.1 Closed-Form Solution for Diffusion Models without Liquidity Shocks

While in general the model solution features four regions: (1.) a gain-realization, (2.) a loss-realization, (3.) a normal holding region, and (4.) a deep-loss holding region, only the first three regions are reached on the optimal path in our diffusion model as in IJ (2013) and HY (2019). This is because a.) as soon as the investor realizes losses or gains, he resets x to one and returns to the normal holding region; and b.) the stock prices follow diffusion processes, which are continuous.

Since we focus on the optimal path, it is convenient to use the heuristic real-option approach as in IJ (2013), which correctly identifies the three regions on the optimal path.[37] Smooth-pasting conditions also correctly characterize the optimal realization strategies in IJ (2013), as pointed out by HY (2019).

In the (normal) holding region, we conjecture and later verify the following closed-form expression for $v(w,x)$:

$$v(w,x) = C_1(w)x^{\eta_1} + C_2(w)x^{\eta_2}, \tag{B.1}$$

where $\eta_1 > 0$ and $\eta_2 < 0$ are the two roots of the fundamental quadratic equation:[38]

$$h(\eta) = \frac{\sigma^2}{2}\eta(\eta-1) + (\mu - r)\eta - \delta_e\,. \tag{B.2}$$

Next, we determine $C_1(w)$ and $C_2(w)$ as functions of w.[39]

We show that the normal holding region is characterized by two endogenous threshold functions, $\underline{x}(w)$ and $\overline{x}(w)$, satisfying $\max\{-w/\kappa,\,0\} \le \underline{x}(w) < 1 \le \overline{x}(w)$, where $-w/\kappa$ is the involuntary liquidation threshold implied by the leverage constraint (6) when $w < 0$ (the leverage case). That is, the investor optimally keeps his scaled savings constant at w in the $x \in (\underline{x}(w), \overline{x}(w))$ region and trades only when $x = \underline{x}(w)$ or $x = \overline{x}(w)$. We refer to $\overline{x}(w)$ and $\underline{x}(w)$ as the optimal gain- and loss-realization boundary, respectively.

Using (23) and (B.1), we obtain the following expression for the scaled value function $f(w,x)$ in both the gain- and loss- realization regions:

$$f(w,x) = u\Big((1-\theta_s)x - 1\Big) + \frac{C_1(w^*) + C_2(w^*)}{[w^* + (1+\theta_p)]^\beta}\,[w + (1-\theta_s)x]^\beta\ , \tag{B.3}$$

where w^* is the optimal post-realization scaled savings given by:

$$w^* = \arg\max_{\widehat{w} \ge -\kappa} \frac{C_1(\widehat{w}) + C_2(\widehat{w})}{[\widehat{w} + (1+\theta_p)]^\beta}\,. \tag{B.4}$$

There are two scenarios for the solution. First if the investor is willing to voluntarily realize losses, i.e., when $\underline{x}(w) > \max\{-w/\kappa,\,0\}$ holds, then the two realization boundaries, $\overline{x}(w)$ and $\underline{x}(w)$, satisfy the following value-matching and the smooth-pasting

[37] By the heuristic real-option approach, we refer to the following commonly used solution method in the real-options literature. First, conjecture that the value function satisfies an HJB equation in the holding (waiting) region. Second, specify the payoff functions in the realization (exercise) region. Finally, impose the value-matching and smooth-pasting conditions. In Internet Appendix IB, we show that while intuitive and easy to use compared with the variational inequality, the heuristic real-option approach may give a wrong solution in a jump-diffusion model. Researchers shall exercise caution when applying the heuristic approach to solve real-option problems.

[38] Different from the classical real-options literature, e.g., McDonald and Siegel (1986) and Dixit and Pindyck (1994), the positive root η_1 may be less than one, which means that the value function may not be globally convex in X. See Figure 3 for an example.

[39] Recall that w_t is constant over time.

conditions:

$$
\begin{aligned}
C_1(w)[\overline{x}(w)]^{\eta_1} + C_2(w)[\overline{x}(w)]^{\eta_2} &= f(w, \overline{x}(w)), & \text{(B.5)}\\
C_1(w)[\underline{x}(w)]^{\eta_1} + C_2(w)[\underline{x}(w)]^{\eta_2} &= f(w, \underline{x}(w)), & \text{(B.6)}\\
C_1(w)\eta_1[\overline{x}(w)]^{\eta_1-1} + C_2(w)\eta_2[\overline{x}(w)]^{\eta_2-1} &= f_x(w, \overline{x}(w)), & \text{(B.7)}\\
C_1(w)\eta_1[\underline{x}(w)]^{\eta_1-1} + C_2(w)\eta_2[\underline{x}(w)]^{\eta_2-1} &= f_x(w, \underline{x}(w)), & \text{(B.8)}
\end{aligned}
$$

where $f(w, x)$ is given in (B.3). Second, if the investor realizes losses only when the leverage constraint (6) binds, we replace the smooth-pasting condition (B.8) with $\underline{x}(w) = \max\{0, -w^*/\kappa\}$ implied by (6).

Next, we analyze the $\kappa > 0$ case and then turn to the no-leverage case where $\kappa = 0$.

B.1.1 The $\kappa > 0$ Case

First, we solve for the five numbers: w^*, $C_1(w^*)$, $C_2(w^*)$, $\overline{x}(w^*)$, and $\underline{x}(w^*)$, using the five-equation system (B.4)-(B.8). Then, we substitute the values of w^*, $C_1(w^*)$, and $C_2(w^*)$ into (B.3) to obtain $f(w, x)$. Finally, we solve for the four functions: $C_1(w)$, $C_2(w)$, $\overline{x}(w)$, and $\underline{x}(w)$ by substituting $f(w, x)$ into the four-equation system (B.5)-(B.8).

Step 1: Show $w^* > -\kappa$. The optimal new (scaled) allocation to the risk-free asset w^* when he trades must be in the interior region of w: $w^* > -\kappa$. Otherwise, immediately after trading, the investor has to incur trading costs again, which is suboptimal. Therefore, the following first-order condition for (B.4) implies:[40]

$$[w^* + 1 + \theta_p][C_1'(w^*) + C_2'(w^*)] - \beta[C_1(w^*) + C_2(w^*)] = 0\,. \tag{B.9}$$

We solve for w^* and the four functions $C_1(w), C_2(w), \underline{x}(w), \overline{x}(w)$ in two steps.

Step 2: Solve for the five numbers: w^*, $C_1(w^*)$, $C_2(w^*)$, $\overline{x}(w^*)$, and $\underline{x}(w^*)$. Recall that $w_s = w^*$ is absorbing in that for all $s \geq t$ where $t = \{u : \inf_u w_u = w^*\}$. Conditional on $w = w^*$, the solution boils down to a one-dimensional problem and there are two possible scenarios, as we show next.

- Scenario (i) where the leverage constraint does not bind: $\underline{x}(w^*) > \max\{0, -w^*/\kappa\}$. In this case, the value-matching and smooth-pasting conditions hold at both $x = \underline{x}(w^*)$ and $x = \overline{x}(w^*)$. We thus obtain a candidate solution for these five numbers by solving a system of five equations: (B.5)-(B.9).

[40] We verify that the second-order condition holds at w^*, i.e., for $m''(w^*) < 0$.

- Scenario (ii) where the leverage constraint binds: $\underline{x}(w^*) = \max\{0, -w^*/\kappa\}$. We then obtain a candidate solution for $\{w^*, C_1(w^*), C_2(w^*), \overline{x}(w^*)\}$ by solving a system of four equations: (B.5)-(B.6) and (B.9), as $\underline{x}(w^*) = \max\{0, -w^*/\kappa\}$.

Step 3: Solve for the four functions $\{C_1(w), C_2(w), \underline{x}(w), \overline{x}(w)\}$ where $w \neq w^*$.

- For scenario (i) introduced above, using the value-matching and smooth-pasting conditions at $x = \underline{x}(w)$ and $x = \overline{x}(w)$, we derive a candidate solution for $\{C_1(w), C_2(w), \underline{x}(w), \overline{x}(w)\}$ by solving a system of four equations: (B.5)-(B.8). Note that we can use the explicit expression for the payoff function $f(w, x)$, obtained from our analysis from Step 2 (for $w = w^*$).

- For scenario (ii) introduced above, we obtain the candidate solution for $\{C_1(w), C_2(w), \overline{x}(w)\}$ by solving a system of three equations (B.5)-(B.7). This is because $\underline{x}(w) = \max\{-w/\kappa,\ 0\}$ holds as the fourth equation.

Finally, comparing the candidate solutions from the two scenarios and choosing the one that gives the larger value of $v(w^*, 1)$, we obtain the optimal solution.

We now have described the procedure for obtaining the optimal solution for the $\kappa > 0$ case where the investor can borrow. Next, we consider the no-borrowing case: $\kappa = 0$.

B.1.2 The $\kappa = 0$ Case

In this case, since the investor cannot borrow, there is no forced loss realization, which implies $\mathcal{S} = \{x > 0,\ w \geq 0\}$. Since it may be optimal to invest all his budget in the stock he chooses, there are two possible cases for $w = w^*$: (i) the $w^* > 0$ case where the first-order condition (B.9) holds and (ii) the $w^* = 0$ case. For both cases, we have two scenarios depending on whether the investor is willing to voluntarily realize losses: (a) $\underline{x}(w^*) > 0$ and (b) $\underline{x}(w^*) = 0$. Using the same argument as in the $\kappa > 0$ case, we can obtain the optimal solution for w^* and the four functions $\{C_1(w), C_2(w), \underline{x}(w), \overline{x}(w)\}$. .

For the optimal trading strategy and the value function, we provide a verification theorem in Proposition ID.3 and a proof in Internet Appendix **??**. Next, we describe the transition dynamics of (x, w) to the target $(1, w^*)$.

B.2 Transition Dynamics to the Target Position $(x, w) = (1, w^*)$

We analyze the transition dynamics for an investor starting with a position given by (W, X, B) or equivalently in the scaled variables $(x, w) = (X/B, W/B)$. Let $\tau(x, w)$

denote the first time the investor realizes gains or losses starting from (x, w).[41] Immediately after $\tau(x, w)$, the investor adjusts his position to $(x_{\tau+}, w_{\tau+}) = (1, w^*)$. For all $t \geq \tau(x, w)$, $w_t = w^*$ and x_t follows (20) before the investor trades again.

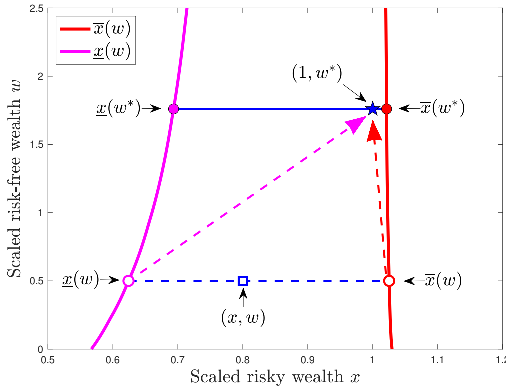

Figure B.1: TRANSITION DYNAMICS OF (x_t, w_t).

Figure B.1 illustrates the transition dynamics.[42] First, starting from the position (x, w) (the blue square), the investor's scaled risky wealth x_t moves stochastically along the horizontal dashed blue line (because $dw_t = 0$) in response to shocks until x_t reaches either the open pink circle or the open red circle. After immediately realizing losses or gains at $\tau(x, w)$, the investor readjusts his allocation to the blue starred position $(1, w^*)$. Note the discrete change of his scaled risk-free wealth from w to w^* at $\tau(x, w)$.

After $\tau(x, w)$, the investor stays on the horizontal dashed blue line as $w_t = w^*$ for all t. The scaled risky wealth x evolves stochastically until either the loss-realization boundary $\underline{x}(w^*)$ or the gain-realization boundary $\overline{x}(w^*)$ is reached. Afterward reaching either boundary line, the investor again readjusts his position to $(1, w^*)$.

Next, we propose a numerical scheme for our jump-diffusion model.

B.3 Numerical Scheme: Penalty Method

For the generalized jump-diffusion model, we do not have closed-from solutions. We obtain the solution by working with the following variational inequality:

$$\max\{\mathscr{L}^{\mathcal{J}} v(w, x),\ f(w, x) - v(w, x)\} = 0 \quad \text{for } x \geq 0,\ w \geq 0\ , \tag{B.10}$$

[41] Mathematically, given the optimal double-barrier policy, $\overline{x}(w)$ and $\underline{x}(w)$, because $dw_t = 0$, we have $\tau(x, w) = \inf\{t \geq 0 \mid x_0 = x,\ x_t \notin (\underline{x}(w), \overline{x}(w))\}$.

[42] We start the dynamics in the normal holding region. Parameter values are from our quantitative analysis in Table 2 in Section 4.

where $f(w,x)$ and $\mathscr{L}^{\mathcal{J}}$ are respectively given by

$$f(w,x) = u\big((1-\theta_s)x - 1\big) + \max_{\widehat{w}\geq 0} \frac{v(\widehat{w},1)}{(\widehat{w}+1+\theta_p)^{\beta}} \left[(1-\theta_s)x + w\right]^{\beta} \ , \quad \text{(B.11)}$$

$$\mathscr{L}^{\mathcal{J}} v(w,x) = \frac{\sigma^2 x^2}{2} v_{xx}(w,x) + (\mu - r)x v_x(w,x) - \delta_e v(w,x) + \rho \left(\mathbb{E}[v(w,Yx)] - v(w,x)\right) \ . \quad \text{(B.12)}$$

We use the following iteration algorithm developed by Dai and Zhong (2010).

Step 1. For $k \geq 0$, calculate the payoff function $f^{(k)}(w,x)$ in (B.11) taking the k-th solution $v^{(k)}(w,x)$ as given (we start the iteration with $v^{(0)}(w,x) = 1$).

Step 2. Update $v^{(k+1)}(w,x)$ by solving the following equation with a penalty term:

$$0 = \frac{1}{2}\sigma^2 x^2 v_{xx}^{(k+1)}(w,x) + (\mu - r)x v_x^{(k+1)}(w,x) - (\delta_e + \rho)v(w,x) + \rho\, \mathbb{E}[v^{(k)}(w,Yx)] + P \times \mathbf{1}_{\{f^{(k)}(w,x) - v^{(k)}(w,x) > 0\}} \left(f^{(k)}(w,x) - v^{(k+1)}(w,x)\right) \ ,$$

where P is a large penalty constant, e.g., $P = 10^6$.

Step 3. Stop iteration if the relative error is less than a given tolerance level ε, e.g., $\varepsilon = 10^{-9}$:

$$\frac{||v^{(k+1)} - v^{(k)}||}{\max\{1, ||v^{(k)}||\}} < \varepsilon \ .$$

Otherwise, go to Step 1 and continue the iterative process.

C Value of Saving in the Risk-free Asset and Leverage

To measure the value created by the option to save in the risk-free asset and/or to use leverage, we propose a certainty-equivalent-wealth-based measure.

First, we report the investor's value function in models of BX (2012) and IJ (2013), where the investor can only invest either in a stock or in the risk-free asset. We focus on the case where the voluntary participation condition (to invest in stocks) is satisfied.

Let $V_N(X_t, B_t)$ denote the investor's value function at t, where the subscript N refers to the "no-option" benchmark case. As $V_N(X,B)$ is homogeneous in X and B with degree β, it is convenient to work with the scaled value function:

$$v_N(x) = V_N(x,1) = B^{-\beta} V_N(X,B) \quad \text{(C.1)}$$

and the corresponding scaled payoff function: $f_N(x) = B^{-\beta} F_N(X, B)$, where $x = X/B$. When trading the stock at τ, the risky wealth at $\tau+$ is given by:

$$X_{\tau+} = \frac{1-\theta_s}{1+\theta_p} X_\tau \,. \tag{C.2}$$

Second, we ask the following question for the investor with an initial budget of Π_0 in our model. How much compensation (fraction Δ of his budget Π_0) does this investor require for him to permanently forgo the option to save in the risk-free asset and use leverage? That is, for this investor to be indifferent between (a.) living in our model and (b.) living in the economy analyzed by BX (2012) and IJ (2013) but with a (higher) budget of $(1+\Delta)\Pi_0$, the following indifference condition that defines Δ must hold:

$$V(W_{0+}, X_{0+}, X_{0+}) = V_N\left(\frac{(1+\Delta)\Pi_0}{1+\theta_p}, \frac{(1+\Delta)\Pi_0}{1+\theta_p}\right), \tag{C.3}$$

where $W_{0+} = \Pi_0 - (1+\theta_p)X_{0+}$ is the risk-free wealth after the investor purchases the stock. Using the homogeneity property, we obtain the following expression for $\Delta(w)$:

$$\Delta(w) = \left(\frac{m(w)}{m_N}\right)^{1/\beta} - 1\,, \tag{C.4}$$

where $w = W_{0+}/X_{0+}$, $m(\,\cdot\,)$ is defined in (25), and $m_N = v_N(1)/(1+\theta_p)^\beta$.

Since $\Delta(w)$ is increasing with $m(w)$ and $w = w^*$ is the maximand of $m(w)$, the measure $\Delta(w)$ is also maximized when $w = w^*$.

D Deep-Loss Region

In this appendix, we explain why it is possible for the investor in our model to voluntarily sell a stock in deep losses while it is not possible in IJ (2013) and HY (2019). We show this difference between the two models by focusing on the payoff functions near $x = 0$.

Predictions of IJ (2013) and HY (2019). We first show that if the investor has to make a binary choice between a stock and the risk-free asset as in IJ (2013) and HY (2019), it is not possible for the investor to sell a stock in deep losses.[43] Recall that the scaled payoff function in IJ (2013) is (in our notations):

$$f_N(x) = u((1-\theta_s)x - 1) + \left(\frac{1-\theta_s}{1+\theta_p}x\right)^\beta v_N(1)\,,$$

[43] HY (2019) provide a proof of this result.

where $v_N(x)$ is given in (C.1). In the deep-loss region, i.e., when x approaches zero,

$$\lim_{x\to 0} f_N(x) = \lim_{x\to 0} u((1-\theta_s)x - 1) + \left(\frac{1-\theta_s}{1+\theta_p}x\right)^\beta v_N(1) = -\lambda < 0.$$

Therefore, voluntarily realizing deep losses, which yields negative utility, must be suboptimal in IJ (2013) and HY (2019). This is because the investor can always achieve zero realization utility by never realizing losses. In sum, being passive is optimal if the stock he owns is in a deep loss because the entire budget is invested in a single stock.

Also note that this no-deep-loss-realization result remains valid when the solution only features two regions. In that case, realizing losses is so painful that the investor does not want to realize losses of any size as in BX (2012). This two-region-solution case is a special case of our four-region solution analyzed above.

Predictions of Our Model. Now, we return to our model and explain why there are two cases, a and b, that generate quite different economic predictions when an investor has two-layered mental accounts and does not have to invest his entire mental budget in a single stock.

a. The solution features three regions. Thus, realizing deep losses is optimal. This is a new key result of our model, summarized in the first row for "our model" in Table 1.

b. The solution features four regions. Therefore, realizing deep losses is not optimal as in IJ (2013) and HY (2019). This case generates sell-after-rebound. The second row for "our model" in Table 1 describes this case.

Next, we answer "which case gives the solution under what conditions" and provide intuition. Recall that in our model, the investor's scaled payoff function is

$$f(w^*, x) = u((1-\theta_s)x - 1) + \left(\frac{w^* + (1-\theta_s)x}{w^* + 1 + \theta_p}\right)^\beta v(w^*, 1).$$

It is helpful to first consider the limit of $f(w^*, x)$ for the $w^* > 0$ case:

$$\begin{aligned}\lim_{x\to 0} f(w^*, x) &= \lim_{x\to 0}\left[u((1-\theta_s)x - 1) + \left(\frac{w^* + (1-\theta_s)x}{w^* + 1 + \theta_p}\right)^\beta v(w^*, 1)\right] \\ &= -\lambda + \left(\frac{w^*}{w^* + 1 + \theta_p}\right)^\beta v(w^*, 1). \qquad \text{(D.1)}\end{aligned}$$

Equation (D.1) implies the following two possibilities.

a. For w^* that is large enough so that $f(w^*, 0) > 0$, (e.g., when compared with λ), it is optimal for the investor to voluntarily realize deep losses, as doing so yields a higher value $v(w^*, 0)$ than being permanently passive, which yields zero utility. This is our case *a* where the solution features three regions. Being able to save a fraction of his trading budget for future potential gain realizations in our model is the key force driving this result.

b. For w^* that is sufficiently close to zero, the scaled payoff is negative: $f(w^*, x) < 0$. Because the investor can always achieve zero realization utility by never realizing losses, realizing deep losses cannot be optimal (as $f(w^*, x) < 0$). This is why being passive is optimal in the deep-loss region.[44] This is our case *b* where the solution features four regions as in IJ (2013) and HY (2019).

In sum, if $w^* > 0$, the solution features either three regions (case *a*) or four regions (case *b*) depending on how large w^* is. Finally, if it is optimal to use leverage ($w^* < 0$), the investor has no option but to realize losses when the scaled risky wealth x approaches the involuntary realization boundary: $-w^*/\kappa > 0$ implied by the leverage constraint (Otherwise, lenders cannot break even and the investor cannot borrow *ex ante*).

E Comparative Statics

In this appendix, we conduct comparative statics analyses for optimal policies: the steady-state value w^* and trading strategies $(\underline{x}^*, \overline{x}^*)$. We focus on the effects of liquidity shocks, the investment opportunity, and the investor's preferences.

E.1 Effects of Liquidity Shocks: ξ

In this subsection, we consider the effect of liquidity shocks on the investor's risk taking and value function curvature. We show that liquidity shocks provide discipline on the investor's option to sit out and encourage risk taking. However, the effects of the liquidity shocks on the value function curvature also critically depend on whether the leverage constraint (6) binds or not.

When the Leverage Constraint (6) Does Not Bind. In Figure E.1, we plot w^* and the value function curvature measure, RAM $\gamma(1)$, as we vary the liquidity shock

[44] In IJ (2013) and HY (2019), $f(w^*, 0) = -\lambda < 0 \leq v(w^*, 0)$ holds automatically.

arrival rate ξ, for an S-shaped utility with $\alpha_+ = \alpha_- = 0.5$, $\lambda = 1.5$, and $\beta = 0.3$. Panel A of Figure E.1 shows that as the liquidity shock becomes more frequent (as ξ increases), the investor increases his stockholdings by decreasing w^*. This is because liquidity shocks effectively shorten the investment horizon and make savings (in the risk-free asset), which yield no utility bursts, more costly. The effect of liquidity shocks on w^* is substantial: Increasing the liquidity shock arrival rate from $\xi = 0$ to $\xi = 0.5$ causes the investor's stockholdings to increase by 5.4 times from 36% of his total trading budget to 196% (as w^* decreases from 1.76 to -0.5). Note that w^* turns negative for $\xi \geq 0.15$, which means that the investor uses leverage to amplify his stockholdings when the arrival rate is sufficiently high ($\xi \geq 0.15$). We verify that the leverage constraint (6) (for Figure E.1) does not bind by checking $\underline{x}^* > \max\{0, -w^*/\kappa\}$.

Panel B of Figure E.1 shows that the investor's value function curvature, measured by RAM $\gamma(1)$, is negative and its absolute value $\gamma(1)$ increases as ξ increases. This means that the investor is risk seeking and becomes more willing to do so, as ξ increases. This is because liquidity shocks provide discipline on the investor's option to sit out and increase the convexity of the investor's value function: $\gamma(1)$.

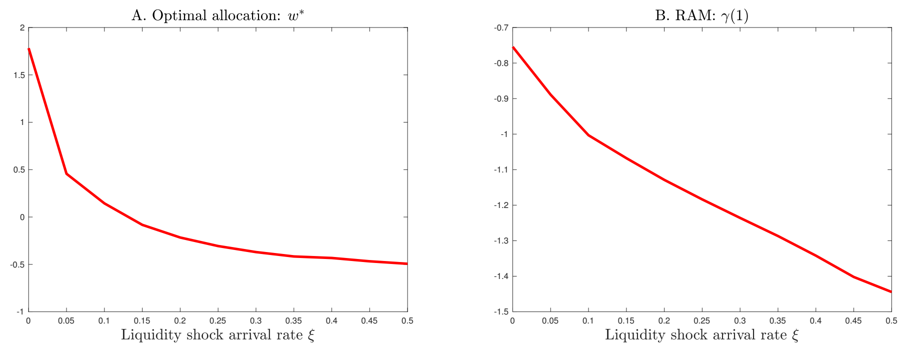

Figure E.1: Effects of liquidity shocks on w^* and value function curvature: RAM $\gamma(1)$ when leverage constraint (6) does not bind. Panels A and B plot the optimal allocation to the risk-free asset w^* and RAM $\gamma(1)$, respectively, as we vary the liquidity shock arrival rate ξ. As ξ increases, the investor increases his stockholdings (w^* decreases) and becomes more willing to take risk. See Table 2 for parameter values.

When the Leverage Constraint (6) Binds. In Figure E.2, we plot w^* and RAM $\gamma(1)$ for a case where the investor has a piecewise linear realization utility as in BX (2012). That is, we set $\alpha_+ = \alpha_- = \beta = 1$, loss aversion parameter at $\lambda = 1.5$, $\delta = 0.35$, and keep all the other parameters the same as in Table 2. Panel A of Figure E.2 confirms that the main result still holds, i.e., the investor's stock allocation increases with the arrival rate ξ. This is because liquidity shocks provide discipline on the investor's option to sit out and encourage him to take risk.

However, the investor's value function curvature measure, RAM $\gamma(1)$, behaves quite differently from the case analyzed in Figure E.1. Rather than decreasing with ξ, RAM $\gamma(1)$ increases with ξ because the leverage constraint (6) binds for all ξ (Recall our analysis in Section 6 for the case where the investor has a piecewise linear realization utility). The intuition is as follows. When the leverage constraint binds, the investor is forced to realize losses to meet his debt payments, which generates a large utility cost (a negative utility burst). Anticipating this scenario, the investor is endogenously averse to stock return volatility when he is close to the binding leverage constraint. This is an example showing that leverage constraints have a significant effect on the investor's risk taking and can mitigate his disposition effect, as we discussed earlier.

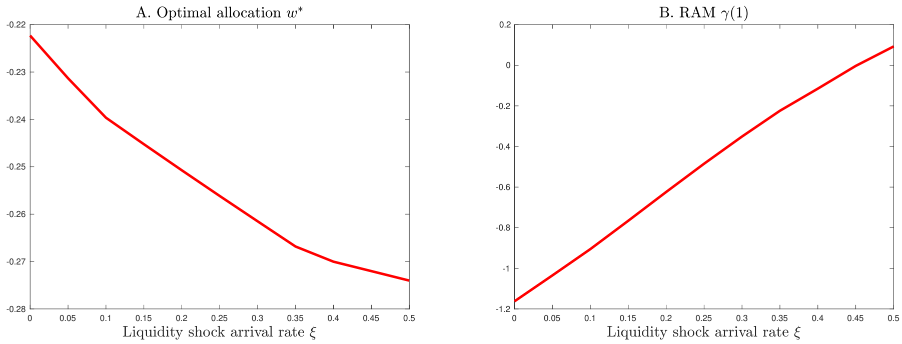

Figure E.2: Effects of liquidity shocks on w^* and value function curvature: RAM $\gamma(1)$ when leverage constraint (6) binds. Panels A and B plot the optimal allocation to the risk-free asset w^* and RAM $\gamma(1)$, respectively, as we vary the liquidity shock arrival rate ξ. As ξ increases, the investor increases his stockholdings (w^* decreases) and becomes less risk taking due to binding of leverage constraint. Parameter values: $\alpha_{\pm} = \beta = 1$, $\delta = 0.35$, and other parameter values are listed in Table 2.

E.2 Investment Opportunity: (r, μ)

Panels A1 and A2 of Figure E.3 plot the ratio between his risk-free savings and stockholdings, w^*, as we vary the risk-free rate r and the expected stock return μ, respectively. The investor increases his stock allocations by decreasing w^* as μ increases or r decreases.

The quantitative effects of changing the investment opportunity within an economically plausible range are large. What is the effect of decreasing r from 3% to 0? The ratio w^* decreases significantly from 1.76 to -0.14 (Panel A1), which means that the investor's stock allocation increases by 3.2 times from 36% of his budget to a leveraged position with 115% of his budget. Similarly, he increases his stock allocation by decreasing w^*

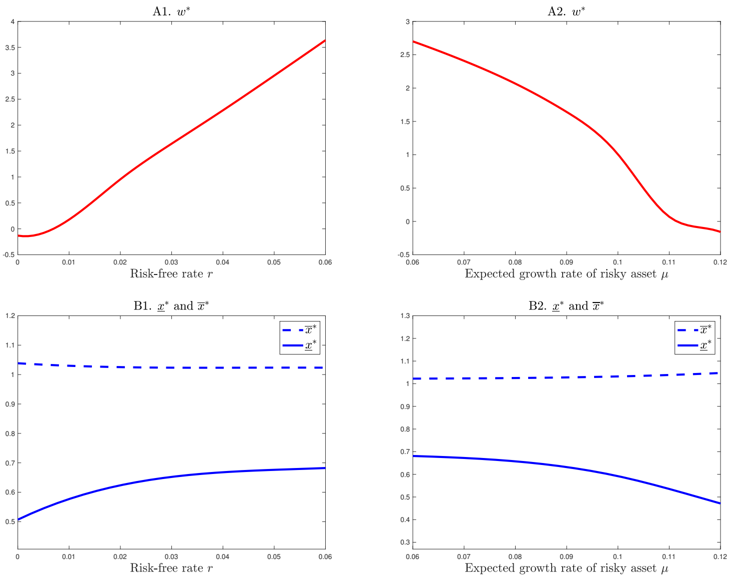

Figure E.3: Comparative Statics – Effects of Changing Investment Opportunity: (r, μ). Panels A1 and A2 plot the optimal w^* as we vary r and μ, respectively. Panels B1 and B2 plot the optimal gain-realization threshold $\overline{x}^*$ (the dashed blue lines) and loss-realization threshold $\underline{x}^*$ (the solid blue lines), as we vary r and μ, respectively. See Table 2 for all other parameter values.

from 1.76 to -0.15, which implies that his stock allocation by 3.2 times from 36% of his budget to a leveraged position with 116% of his budget (Panel A2).

Panels B1 and B2 of Figure E.3 plot the gain-realization threshold $\overline{x}^*$ (dashed blue lines) and the loss-realization threshold $\underline{x}^*$ (solid blue lines), as we vary r and μ, respectively. Recall that as we increase μ or decrease r, the investor increases his dollar exposure to the stock (decreasing w^*), which in turn makes him more reluctant to realize losses and gains. As a result, the holding region $(\underline{x}^*, \overline{x}^*)$ widens as μ increases (Panel B1) or r decreases (Panel B2).

Finally, the quantitative effects of changing r or μ on loss realization (the lower boundary) are much more significant than on gain realization (the upper boundary).

E.3 Realization Utility: $(\lambda, \alpha_-, \alpha_+)$

In Figure E.4, we conduct a comparative static analysis for three key preference parameters: loss aversion λ, loss sensitivity α_-, and gain sensitivity α_+. We plot the optimal allocation, w^*, as functions of λ, α_-, and α_+ in Panels A1, A2, and A3, respectively. Similarly, we plot the gain-realization threshold $\overline{x}^*$ (dashed blue lines) and the loss-realization threshold $\underline{x}^*$ (solid blue lines), as functions of λ, α_-, and α_+ in Panels B1, B2, and B3, respectively.

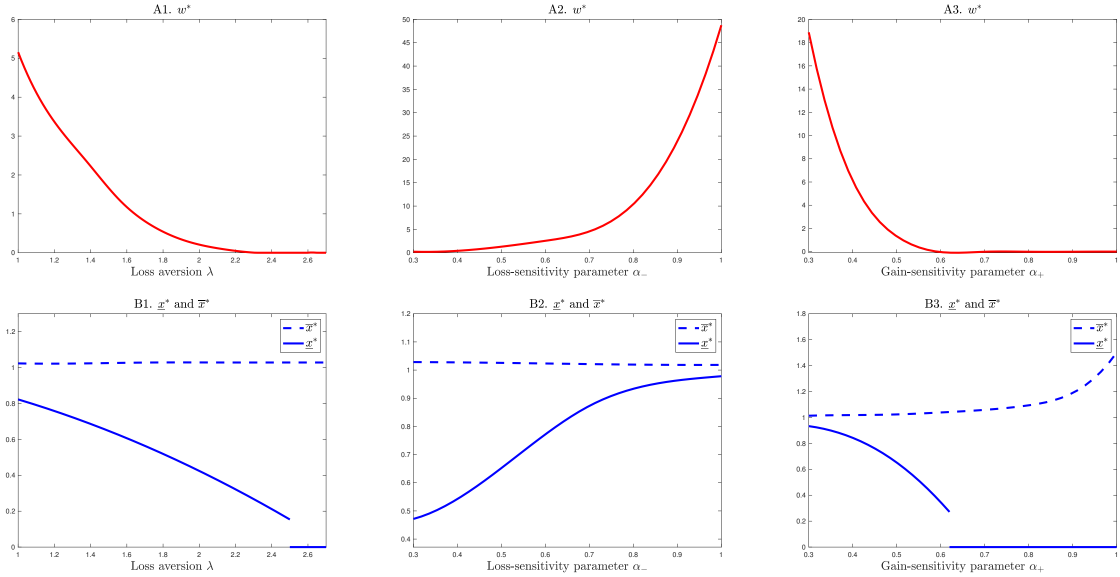

Figure E.4: Comparative Statics: Effects of Changing Realization Utility: $(\lambda, \alpha_-, \alpha_+)$. Panels A1, A2, and A3 plot the optimal w^* as we vary λ, α_- and α_+, respectively. Panels B1, B2, and B3 plot the optimal gain-realization threshold $\overline{x}^*$ (the dashed blue lines) and the optimal loss-realization threshold $\underline{x}^*$ (the solid blue lines), as we vary λ, α_- and α_+, respectively. All parameter values other than the one being studied are reported in Table 2.

To ease comparison with IJ (2013), we first analyze the lower row of Figure E.4. The key takeaways are as follows. First, as the investor becomes more loss averse (higher λ), or more sensitive to losses (lower α_-), or less sensitive to gains (higher α_+), the loss-realization threshold $\underline{x}^*$ decreases and the gain-realization threshold $\overline{x}^*$ increases. The effects on $\underline{x}^*$ and $\overline{x}^*$ reinforce each other widening the holding region. A more loss-averse or a more loss-sensitive investor is less willing to realize losses. A less gain-sensitive (higher α_+) investor waits longer to realize gains reducing the option value of resetting the reference level and lowering the loss-realization threshold $\underline{x}^*$.

Second, comparing the loss-realization threshold $\underline{x}^*$ (dashed lines) with the gain-realization threshold $\overline{x}^*$ (solid lines), we clearly see that the quantitative effects of changing these preference parameters on $\underline{x}^*$ are much more significant than on $\overline{x}^*$. For loss aversion satisfying $\lambda \geq 2.5$ or gain sensitivity satisfying $\alpha_+ \geq 0.62$, the investor never realizes losses. These results are similar to those in IJ (2013).

The upper row shows that w^* decreases with loss aversion λ (Panel A1), increases with α_- (Panel A2), and decreases with α_+ (Panel A3), reinforcing the results discussed above for the lower row. Recall that as we decrease loss aversion (reducing λ), reduce loss sensitivity (increasing α_-), and increase gain sensitivity (decreasing α_+), the investor trades more frequently (a narrower holding region $(\underline{x}^*, \overline{x}^*)$). As a result, the benefit of placing more frequent and smaller stock trades causing w^* to increase.

Quantitatively, the effects of changing these preference parameters within economically relevant ranges on w^* are large. For example, the investor's stockholdings will increase from 64% to 100% of his total mental trading budget (caused by a decrease of w^* from 1.76 to zero) if any one of the following three changes takes place: a) loss aversion λ increases from 1.5 to 2.5, b) the parameter α_- decreases from 0.5 to 0.3, and c) the parameter α_+ increases from 0.5 to 0.62.

In sum, the option to save a fraction of his mental budget or use leverage significantly alters the investor's dynamic trading strategies and has large quantitative effects.

We next analyze the effect of liquidity shocks on w^* and the value function curvature.

F Proof of Proposition 1 (Piecewise Linear Realization Utility)

We prove this proposition in two steps. First, we show that the investor never realizes losses as in BX (2012) so that $\underline{x}(w^*) = \max\{-w^*/\kappa, 0\}$. Second, we show that the investor has no incentives to save: $w^* \leq 0$, despite having the option to do so.

Step 1: Prove $\underline{x}(w^*) = \max\{-w^*/\kappa, 0\}$. First, for piecewise linear $u(\cdot)$ where $\alpha_\pm = \beta = 1$, $f(w^*, x)$ is given by

$$f(w^*, x) = \begin{cases} (1-\theta_s)x - 1 + \frac{v(w^*,1)}{w^*+1+\theta_p}\big(w^* + (1-\theta_s)x\big) & \text{if } x \in [1/(1-\theta_s), \infty) \\ -\lambda\big(1-(1-\theta_s)x\big) + \frac{v(w^*,1)}{w^*+1+\theta_p}\big(w^* + (1-\theta_s)x\big) & \text{if } x \in [0, 1/(1-\theta_s)). \end{cases}$$

Note that $f(w^*, x)$ is linear in x with slope $\big(\frac{v(w^*,1)}{w^*+1+\theta_p} + \lambda\big)(1-\theta_s)$ in the region where $x \in [1/(1-\theta_s), \infty)$ and $f(w^*, x)$ is linear in x with slope $\big(\frac{v(w^*,1)}{w^*+1+\theta_p} + 1\big)(1-\theta_s)$ in the region where $x \in [0, 1/(1-\theta_s))$. Because $\lambda \geq 1$, the slope in the loss region where $x \in [0, 1/(1-\theta_s))$ is larger than that in the gain region where $x \in [1/(1-\theta_s), \infty)$. Also, $f(w^*, x)$ is increasing and globally concave in x.

We prove $\underline{x}(w^*) = \max\{-w^*/\kappa, 0\}$ by contradiction. Suppose that the opposite holds in that $\max\{-w^*/\kappa, 0\} < \underline{x}(w^*) < \overline{x}(w^*)$. Then, the optimal trading thresholds are interior, implying that the smooth-pasting conditions apply at both $\underline{x}(w^*)$ and $\overline{x}(w^*)$. Using these conditions together with the concavity of $f(w^*, x)$ in x, we obtain

$$0 < v_x(w^*, \overline{x}(w^*)) = f_x(w^*, \overline{x}(w^*)) \leq f_x(w^*, \underline{x}(w^*)) = v_x(w^*, \underline{x}(w^*)).$$

Using the smooth-pasting conditions at the two boundaries and the result that $v > f$ in the holding region, we know that v_x must attain a local maximum at a point denoted by $\breve{x}$ between the two boundaries: $\breve{x} \in (\underline{x}(w^*), \overline{x}(w^*))$. That is, $v_{xx}(w^*, \breve{x}) = 0 \geq v_{xxx}(w^*, \breve{x})$.

As $\mathscr{L}v(w^*, x) = 0$ in the holding region, differentiating $\mathscr{L}v(w^*, x) = 0$ with respect to x, we obtain the following contradiction at $(w, x) = (w^*, \breve{x})$:[45]

$$0 = \frac{\sigma^2 x^2}{2} v_{xxx} + (\mu - r + \sigma^2) x v_{xx} - (\delta - \mu) v_x \leq -(\delta - \mu) v_x < 0\,, \tag{F.1}$$

where the first inequality uses $v_x(w^*, \breve{x})$ being a local maximum and the last equality follows from the monotonicity condition $v_x > 0$ and the transversality condition: $\delta > \mu$. Therefore, $\underline{x}(w^*) = \max\{-w^*/\kappa,\, 0\}$ and the investor never voluntarily realizes losses.

Step 2: Prove $w^* \leq 0$. To prove this result, it is equivalent to proving that the investor always chooses zero savings ($w^* = 0$) even if he can save in the risk-free asset.

Recall that the general solution of $v(w, x)$ in the holding region is

$$v(w, x) = C_1(w) x^{\eta_1} + C_2(w) x^{\eta_2}\ , \tag{F.2}$$

where $C_1(\,\cdot\,)$ and $C_2(\,\cdot\,)$ are two functions of w to be determined, and $\eta_1 > 0$ and $\eta_2 < 0$ are the two roots of (B.2). Next we prove $w^* \leq 0$ by contradiction.

Suppose that $w^* > 0$. Using $\underline{x}(w^*) = \max\{-w^*/\kappa,\, 0\} = 0$ obtained from Step 1 and $\eta_2 < 0$, we obtain $C_2(w) = 0$. Using the first-order condition for w^*, we obtain

$$C_1'(w^*)(w^* + 1 + \theta_p) = C_1(w^*)\,. \tag{F.3}$$

The value-matching and smooth-pasting conditions at $x = \overline{x}(w^*)$ imply:

$$C_1'(w^*)\big(\overline{x}(w^*)\big)^{\eta_1} = \frac{C_1(w^*)}{w^* + 1 + \theta_p}\ . \tag{F.4}$$

Combining (F.3) and (F.4), we obtain the following result:

$$\overline{x}(w^*) = 1\,, \tag{F.5}$$

which implies that the investor sells immediately after buying a new stock. This is clearly suboptimal as there are no gains and the investor continuously pays transaction costs. Therefore, the investor chooses not to save ($w^* = 0$) even if he can save.

In sum, an investor with piecewise linear realization utility (and loss aversion) has no incentives to save but may use leverage (so that $w^* \leq 0$) when he has access to the risk-free asset.

[45] In the holding region, the coefficients of the differential equation are constant. The classic regularity theory for elliptic equations implies that the solution is infinitely smooth in the holding region (Evans, 2010).

G Stationary Distributions of x in Jump-Diffusion Models

In this appendix, we characterize the stationary distribution of x.

Given the trading policy, characterized by the optimal w^* and the four regions: $\mathcal{H}_d = (0, \underline{x}^{**})$, $\mathcal{R}_- = [\underline{x}^{**}, \underline{x}^*]$, $\mathcal{H}_n = (\underline{x}^*, \overline{x}^*)$, and $\mathcal{R}_+ = [\overline{x}^*, \infty)$, where $0 \le \underline{x}^{**} \le \underline{x}^* < 1 < \overline{x}^*$, the density function $\varphi(\,\cdot\,)$ for the stationary distribution of x for our jump-diffusion model satisfies the Kolmogorov forward equation:

$$\mathcal{K}^*\varphi(x) = 0\,, \tag{G.1}$$

in the regions where $x \in \mathcal{H}\backslash\{1\} = \mathcal{H}_d \cup \mathcal{H}_n\backslash\{1\}$ and $\mathcal{K}^*$ is the operator defined by

$$\mathcal{K}^*\varphi(x) = \frac{d^2}{dx^2}\Big(\frac{1}{2}\sigma^2 x^2 \varphi(x)\Big) - \frac{d}{dx}\Big((\mu - r)x\,\varphi(x)\Big) + \rho\,\left(\mathbb{E}\Big[\frac{\varphi(x/Y)}{Y}\Big] - \varphi(x)\right)\,. \tag{G.2}$$

Note that (G.1) holds for the two holding regions excluding the $x = 1$ point.[46] The last term in (G.2) captures the effect of jumps on φ. Next, we generalize the duration analysis for diffusion models in IJ (2013) to allow for jumps.

Duration of Investment Episodes. Let τ denote the calendar time that the investor realizes the next trading gain or loss. Let $D(x_t)$ denote the expectation of $(\tau - t)$ conditional on the value of x_t at current time t, i.e., $D(x) = \mathbb{E}_t[(\tau - t)|x_t = x]$. The following result holds in the two holding regions where $x \in \mathcal{H} = \mathcal{H}_d \cup \mathcal{H}_n$:

$$\mathcal{K}D(x) = -1\,, \tag{G.3}$$

where $\mathcal{K}$ is the operator, which is adjoint to $\mathcal{K}^*$ defined in (G.2), defined by

$$\mathcal{K}D(x) = \frac{1}{2}\sigma^2 x^2 D_{xx}(x) + (\mu - r)xD_x(x) + \rho\,\left(\mathbb{E}[D(Yx)] - D(x)\right). \tag{G.4}$$

By definition, $D(x) = 0$ if the investor immediately realizes gains or losses: $x \in \mathcal{R}_- \cup \mathcal{R}_+$.

Let Φ_G denote the fractions of time that the stock has unrealized gains (on paper). Using the stationary distribution $\varphi(x)$, we obtain $\Phi_G = \int_1^{\overline{x}^*} \varphi(x)dx$. The fraction of time that the asset has unrealized losses is therefore given by $\Phi_L = 1 - \Phi_G$. The stock has paper losses either in the deep-loss holding region $\mathcal{H}_d$ or when $\underline{x}_1^* < x < 1$ in the normal holding region $\mathcal{H}_d$. Let $\Phi_{\mathcal{H}_d}$ denote the fraction of time that the investor is in the deep-loss holding region $\mathcal{H}_d$: $\Phi_{\mathcal{H}_d} = \int_0^{\underline{x}^{**}} \varphi(x)dx$. Therefore, $\Phi_L - \Phi_{\mathcal{H}_d}$ is the fraction of time that the investor incurs losses in the normal holding region $\mathcal{H}_n$ with $x \in (\underline{x}^*, 1)$.

[46] The $x = 1$ point in $\mathcal{H}_n$ does not satisfy (G.1) as $\varphi(x)$ is not differentiable there. But $\varphi(x)$ is continuous at $x = 1$: $\lim_{x\to 1^-} \varphi(x) = \lim_{x \to 1^+} \varphi(x)$. Other conditions for $\varphi(x)$ are: a.) the fraction of time spent in the gain- and loss-realization regions is zero: $\varphi(x) = 0$ for $x \in \mathcal{R}_- \cup \mathcal{R}_+$, as the investor immediately resets to $x = 1 \in \mathcal{H}_n$, and b.) the density function $\varphi(x)$ integrates to one: $\int_0^{\overline{x}^*} \varphi(x)dx = 1$.

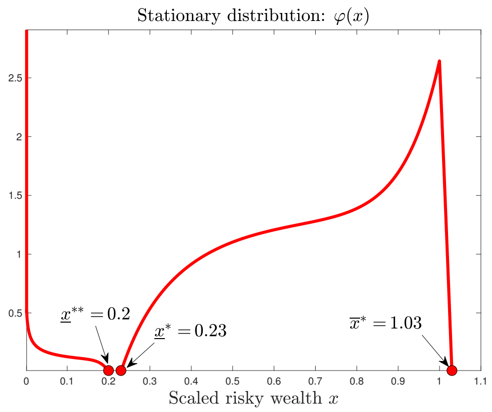

Figure G.1: STATIONARY DENSITY FUNCTION $\varphi(x)$ FOR THE JUMP-DIFFUSION CASE. The long-run probability for $x \in \mathcal{H}_d$ is 3% (the area under the left part of $\varphi(x)$). Parameter values are: $\mu = 19\%$, $\sigma = 20\%$, $\rho = 0.73$, $\psi = 6.3$, $r = 3\%$, $\theta_p = \theta_s = 1\%$, $\alpha_+ = 0.5, \alpha_- = 0.5, \lambda = 2.5$, $\beta = 0.3$, and $\delta = 5\%$.

In Figure G.1, we plot the stationary density function $\varphi(x)$ for x_t for parameter values: $\mu = 19\%$, $\sigma = 20\%$, $\rho = 0.73$, $\psi = 6.3$, $r = 3\%$, $\theta_p = \theta_s = 1\%$, $\alpha_+ = 0.5, \alpha_- = 0.5, \lambda = 2.5$, $\beta = 0.3$, and $\delta = 5\%$, in which case, the optimal $w^* = 0$, $\overline{x}^* = 1.03$, $\underline{x}^* = 0.23$, and $\underline{x}^{**} = 0.2$ such that $\mathcal{H}_d = (0, 0.2)$, $\mathcal{R}_- = [0.2, 0.23]$, $\mathcal{H}_n = (0.23, 1.03)$, and $\mathcal{R}_+ = [1.03, \infty)$.

The area under the left part of the stationary density function $\varphi(x)$ equals $\Phi_{\mathcal{H}_d} = 3\%$, which means that in the long run the investor spends about 3% of his time in the deep-loss holding region $\mathcal{H}_d$. The density function $\varphi(x)$ has a single peak at $x = 0$ in this deep loss region.

The area under the right part of the density function $\varphi(x)$ equals $1 - \Phi_{\mathcal{H}_d} = 97\%$, which means that in the long run the investor spends about 3% of his time in the deep-loss $\mathcal{H}_d$ region and the remaining 97% of his time in the normal holding region $\mathcal{H}_n$. Out of this 97% time spent in $\mathcal{H}_n$, the investor spends about 4% of his entire time in the paper gain region $x \in (1, \overline{x}^*) = (1, 1.03)$, as $\Phi_G = 4\%$, and the other 93% of his time in the normal paper loss region $x \in (\underline{x}^*, 1) = (0.23, 1)$. In the normal holding region $\mathcal{H}_n$, $\varphi(x)$ is single peaked at $x = 1$. This is because the investor can only enter into $\mathcal{H}_n$ from either $\mathcal{R}_+$ (after realizing gains) or $\mathcal{R}_-$ (after realizing losses). For this reason, $\varphi(x)$ is not differentiable at $x = 1$. The expected duration for an investment episode is about 183 days for our baseline jump-diffusion model: $D(1) = 0.5$.

Finally, we compare the two peaks for the density function $\varphi(x)$: one at $x = 0$ and the other at $x = 1$. For the peak at $x = 0$, the density $\varphi(x)$ goes to ∞ because $x = 0$ is an absorbing state. For the peak at $x = 1$, the density $\varphi(x)$ does not go to ∞. Recall that $x = 1$ is the beginning of each investment episode when the investor sells the stock he owns, realizes gains or losses, and then resets his reference level. The differences between the densities at these two peaks, $\varphi(0)$ and $\varphi(1)$, can be seen in Figure G.1.

Internet Appendices: Additional Results and Proofs

In our internet appendices, we provide additional results and proofs.

IA Other Specifications of Reference-Level Dynamics

In this appendix, we generalize our model to allow for a more flexible reference-level process. One implication of this generalization is that our model remains a two-dimensional problem after we use the homogeneity property to simplify the problem. Also, the main qualitative results of our main model in the paper remain valid in generalized formulations. To ease exposition, we turn off liquidity shocks by setting $\xi = 0$.

We assume that the reference level B_t grows at a constant rate ν:

$$dB_t/B_t = \nu\, dt \ ,$$

where ν may differ from r. Then, $x_t = X_t/B_t$ and $w_t = W_t/B_t$ follow

$$\begin{aligned} dx_t/x_t &= (\mu - \nu)dt + \sigma d\mathcal{Z}_t, \\ dw_t/w_t &= (r - \nu)dt \ . \end{aligned}$$

In the hold region, the scaled value function, $v(w, x) = B^{-\beta} V(W, X, B)$, solves

$$\frac{1}{2}\sigma^2 x^2 v_{xx} + (\mu - \nu) x v_x + (r - \nu) w v_w - (\delta - \beta\nu) v = 0 \ , \qquad \text{(IA.1)}$$

for $x > 0$ and $w > -\kappa x$.

Obviously, if $\nu = r$, (IA.1) is the same as the HJB equation (26) in our baseline model with no liquidity shocks. However, if $\nu \neq r$, the optimal trading policy can only be described in the x–w plane rather than on an interval $[\underline{x}(w^*), \overline{x}(x^*)]$. In Panel A of Figure IA.1, we plot the solution using our baseline parameter values (with $\nu = r = 3\%$) in x–w plane. Since $\nu = r$, the optimal trading strategies are characterized by the optimal allocation ratio $w^* = 1.76$ and a holding region defined by a gain-realization threshold $\overline{x}(w^*)$ and a loss-realization threshold $\underline{x}(w^*)$.

In Panel B of Figure IA.1 where $\nu = 6\%$, which is larger than the risk-free rate $r = 3\%$, we plot the solution in the x–w plane while keeping other parameters unchanged. Because of a higher reference-level growth rate ν, the optimal allocation ratio w^* (when the investor trades stocks) is lowered to 1.64. Moreover, the double-barrier trading policy (along the optimal path) is no longer optimal. Specifically, starting from $(x_0, w_0) = (1, w^*)$, the negative growth rate of w_t (as $r - \nu = -3\%$) implies that $w_t < w^*$ during each investing episode. Therefore, the state vector (x_t, w_t) must be in the following holding region:

$$\{(x, w)\,|\,x \in (\underline{x}(w), \overline{x}(w)),\; w \in (-\kappa x, w^*)\}\;, \tag{IA.2}$$

which is inside the area surrounded by the solid red line $\underline{x}(w)$ and the solid blue line $\overline{x}(w)$ as illustrated in Panel B of Figure IA.1.

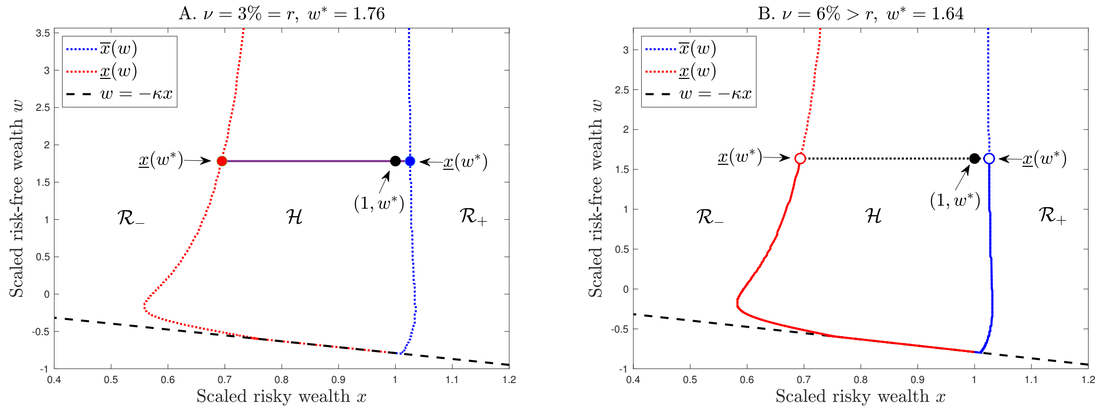

Figure IA.1: Comparing trading strategies under different reference-level growth rates (ν) under optimal paths. In Panel A where $\nu = r = 3\%$, the optimal trading strategies are given by $w^* = 1.76$ and a holding region on a line segment defined by a gain-realization threshold $\overline{x}(w^*) = 1.02$ and a loss-realization threshold $\underline{x}(w^*) = 0.69$. In Panel B where $\nu = 6\% > r = 3\%$, the optimal trading strategies are characterized by $w^* = 1.64$ and the holding region $\mathcal{H}$, which is the area surrounded by the solid lines in the x–w plane. Other parameter values are: $\mu = 9\%$, $\sigma = 30\%$, $\theta_p = \theta_s = 1\%$, $\beta = 0.3$, $\alpha_\pm = 0.5$, $\lambda = 1.5$, $\kappa = 0.79$, and $\delta = 5\%$.

IB Variational Inequality versus Heuristic Real-Option Method

In this appendix, we show how naively applying the smooth-pasting condition might possibly lead to a wrong solution. For simplicity, we consider the case where the investor has no intensive margin, i.e., $w_t^* = 0$ at all t. We continue to work with scaled objects:

$$x = \frac{X}{B}, \quad v(x) = B^{-\beta} V(X, B)\;.$$

Procedure of the Heuristic Real-Option Method. If we had applied the heuristic real-option method to solve the problem defined above, we would proceed as follows.

First, as typically done in the real-options literature, let $\underline{x}^*$ and $\overline{x}^*$ denote the gain- and loss-realization thresholds. Second, we would conjecture that the following HJB equation holds in the $x = (\underline{x}^*, \overline{x}^*)$ holding/waiting region:

$$\mathscr{L}^{\mathcal{J}} v(x) = 0 \tag{IB.1}$$

where $\mathscr{L}^{\mathcal{J}}$ is the operator given by:

$$\mathscr{L}^{\mathcal{J}} v(x) = \frac{1}{2}\sigma^2 x^2 v'' + (\mu - r)xv' - (\delta - \beta r)v + \rho\left(\mathbb{E}[v(xY) - v(x)]\right) . \quad \text{(IB.2)}$$

Third, at the endogenous trading thresholds, $\underline{x}^*$ and $\overline{x}^*$, we would impose value-matching and smooth-pasting conditions as commonly done in the real-options literature:

$$v(x) = f(x) \quad \text{and} \quad v'(x) = f'(x) \quad \text{when } x \in \{\underline{x}^*, \overline{x}^*\} . \quad \text{(IB.3)}$$

Finally, in the gain-realization region where $[\overline{x}^*, \infty)$ and loss-realization regions where $[0, \underline{x}^*]$, we would impose the standard valuation equation (upon stopping):

$$v(x) = f(x) \qquad \text{for } x \in [0, \underline{x}^*] \cup [\overline{x}^*, \infty) . \quad \text{(IB.4)}$$

Let $\widetilde{v}(x)$ and $\widetilde{f}(x)$ denote the value and payoff functions, respectively, obtained by using the above heuristic real-option method.

Having sketched out the procedure for the heuristic real-option method, we next summarize the correct solution method based on a variational inequality. The value function under optimal trading strategies satisfies the following variational inequality:

$$\max\{\mathscr{L}^{\mathcal{J}} v(x),\ f(x) - v(x)\} = 0 \quad \text{for } x \geq 0 , \quad \text{(IB.5)}$$

where $\mathscr{L}^{\mathcal{J}} v(x)$ is given in (IB.2) and $f(x)$ is given in (23).

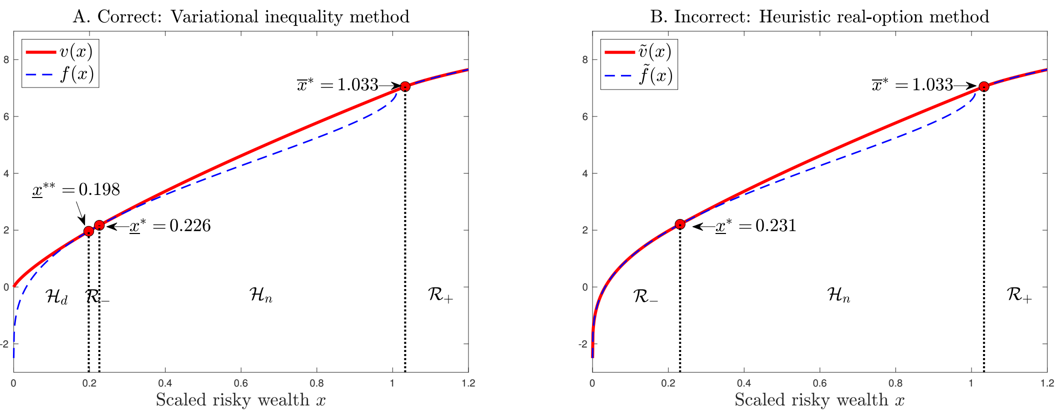

Figure IB.1: VARIATIONAL-INEQUALITY METHOD AND HEURISTIC REAL-OPTION METHOD. Panel A plots $v(x)$ (solid red line) and $f(x)$ (dashed blue line) obtained from the variational-inequality method. There are four regions: deep-loss holding $\mathcal{H}_d$, loss realization $\mathcal{R}_-$, normal holding $\mathcal{H}_n$, and gain realization $\mathcal{R}_+$ regions. $v(1) = 6.8731$. Panel B plots $\tilde{v}(x)$ (solid red line) and $\tilde{f}(x)$ (dashed blue line) obtained from the heuristic real-option method. There are three regions: loss realization $\mathcal{R}_-$, normal holding $\mathcal{H}_n$, and gain realization $\mathcal{R}_+$ regions. $\widetilde{v}(1) = 6.8719$. Parameter values are: $\mu = 19\%$, $\sigma = 20\%$, $\rho = 0.73$, $\psi = 6.3$, $r = 3\%$, $\theta_p = \theta_s = 1\%$, $\alpha_+ = 0.5, \alpha_- = 0.5, \lambda = 2.5$, $\beta = 0.3$, and $\delta = 5\%$.

A Numerical Comparison between Heuristic Real-Option Method and Variational-Inequality Method. Figure IB.1 compares the solutions obtained using the variational-inequality method and the heuristic real-option method. Using the correct variational-inequality method, we obtain four regions: deep-loss holding $\mathcal{H}_d$, loss realization $\mathcal{R}_-$, normal holding $\mathcal{H}_n$, and gain realization $\mathcal{R}_+$ regions. The optimal value is $v(1) = 6.8731$. In contrast, using the heuristic real-option method introduced above, we obtain three regions: loss realization $\mathcal{R}_-$, normal holding $\mathcal{H}_n$, and gain realization $\mathcal{R}_+$ regions. And the optimal value $\widetilde{v}(1) = 6.8719$, which is lower than that by using variational inequality method. This numerical example shows that naively applying the heuristic real-option approach yields a wrong solution.

IC Transversality Conditions

In this appendix, we first provide transversality conditions for jump-diffusion models, then verify transversality conditions (A.7)-(A.9) in Appendix A.2 for diffusion models (which are left out of Appendix A.2 due to space considerations), and finally sketch out the key steps for the verification of transversality conditions for jump-diffusion models.

Transversality Conditions for Jump-diffusion Models. Below we state these conditions:

(i) If $\beta = 1$,

$$\delta > \max\{\mu + \rho\,\mathbb{E}[Y-1],\ 0\}\ . \tag{IC.1}$$

(ii) If $\beta < 1$ and $\frac{\mu - r + \rho\,\mathbb{E}[Y]}{(1-\beta)\sigma^2} \in (0,1)$,

$$\begin{aligned}\delta \quad > \quad & \beta r + \max\Big\{0,\ \frac{\beta(\mu - r + \rho\,\mathbb{E}[Y])^2}{2(1-\beta)\sigma^2}, \\ & \frac{1}{2}\sigma^2\alpha_+(\alpha_+ - 1) + (\mu - r)\alpha_+ + \rho\,\mathbb{E}[Y^{\alpha_+} - 1]\Big\}\,. \end{aligned}\tag{IC.2}$$

(iii) If $\beta < 1$ and $\frac{\mu - r + \rho\,\mathbb{E}[Y]}{(1-\beta)\sigma^2} \notin (0,1)$,

$$\begin{aligned}\delta \quad > \quad & \beta r + \max\Big\{0,\ \frac{1}{2}\sigma^2\beta(\beta-1) + (\mu - r + \rho\,\mathbb{E}[Y])\beta, \\ & \frac{1}{2}\sigma^2\alpha_+(\alpha_+ - 1) + (\mu - r)\alpha_+ + \rho\,\mathbb{E}[Y^{\alpha_+} - 1]\Big\}\,. \end{aligned}\tag{IC.3}$$

Verifying Transversality Conditions for Diffusion Models. Using the proposed conditions (A.7)-(A.9) in Appendix A.2, we show that the value function $V(W, X, B)$ of the problem (14) is finite in two steps.

Step 1. Construct a smooth supersolution of variational inequality (A.1)*.*

Define

$$\Phi(W, X, B) := B^{\beta}\left[C_1\left(\frac{X}{B}\right)^{\alpha_+} + C_2\left(\frac{W}{B} + (1-\theta_s)\frac{X}{B}\right)^{\beta}\right], \tag{IC.4}$$

where C_1 and C_2 are sufficiently large constants. The goal is to show that the function $\Phi(W, X, B)$ defined above is a supersolution to the variational inequality (A.1):

$$\max\left\{\mathcal{L}\Phi(W, X, B),\ F_{\Phi}(W, X, B) - \Phi(W, X, B)\right\} \le 0,$$

where $\mathcal{L}$ is defined in (A.2),

$$F_{\Phi}(W, X, B) = U(G, B) + \max_{\widehat{X} \ge -\widehat{W}/\kappa} \Phi(\widehat{W}, \widehat{X}, \widehat{X}),$$

and $G = (1-\theta_s)X - B$ and $\widehat{W} = W + (1-\theta_s)X - (1+\theta_p)\widehat{X}$. Using the homogeneity property and the scaled variables: $x = X/B$ and $w = W/B$, we can show that

$$\phi(w, x) := B^{-\beta}\Phi(W, X, B) = C_1 x^{\alpha_+} + C_2[w + (1-\theta_p)x]^{\beta} \tag{IC.5}$$

is a supersolution of the HJB equation (A.4):

$$\max\left\{\mathscr{L}\phi(w, x),\ f_{\phi}(w, x) - \phi(w, x)\right\} \le 0, \tag{IC.6}$$

where $\mathscr{L}$ is defined by (A.5) and the function $f_{\phi}(w, x)$ is given by

$$f_{\phi}(w, x) = u((1-\theta_s)x - 1) + \max_{\widehat{w} \ge -\kappa} \frac{\phi(\widehat{w}, 1)}{[\widehat{w} + 1 + \theta_p]^{\beta}}[w + (1-\theta_s)x]^{\beta}. \tag{IC.7}$$

Next, we derive (IC.6) in two steps. First, we verify $f_{\phi}(w, x) \le \phi(w, x)$ as follows:

$$\begin{aligned} f_{\phi}(w, x) &= u((1-\theta_s)x - 1) + \max_{\widehat{w} \ge -\kappa} \frac{C_1 + C_2[\widehat{w} + 1 - \theta_p]^{\beta}}{[\widehat{w} + 1 + \theta_p]^{\beta}}[w + (1-\theta_s)x]^{\beta} \\ &\le C_1 x^{\alpha_+} + C_2[w + (1-\theta_s)x]^{\beta} = \phi(w, x), \end{aligned}$$

where the inequality follows from the definitions of (IC.5) and $u(\cdot)$ given in (12). Second, substituting the expression given in (IC.5) for $\phi(w, x)$ into (A.5), we obtain:

$$\begin{aligned} \mathscr{L}\phi(w, x) &= h(\alpha_+)C_1 x^{\alpha_+} + \widehat{h}(z)C_2[w + (1-\theta_s)x]^{\beta} + \xi u\big((1-\theta_s)x - 1\big) \\ &\le [h(\alpha_+)C_1 + \xi]x^{\alpha_+} + \widehat{h}(z)C_2[w + (1-\theta_s)x]^{\beta}, \end{aligned} \tag{IC.8}$$

where $h(\cdot)$ is given in (B.2), $\widehat{h}(\cdot)$ is given by

$$\widehat{h}(z) = \frac{1}{2}\sigma^2\beta(\beta-1)z^2 + (\mu - r)\beta z - (\delta_e + \xi) \ ,$$

and

$$z = \frac{(1-\theta_s)x}{w + (1-\theta_s)x} \in [0, K] \ . \tag{IC.9}$$

Equation (IC.9) is implied by the leverage constraint $w \geq -\kappa x$, where K is given in (A.6). Then using the conditions given in (A.7)-(A.9), we can show that $h(\alpha_+) \leq 0$ and $\widehat{h}(z) \leq 0$, which together imply $\mathscr{L}\phi(w,x) \leq 0$. We thus have verified (IC.6).

Step 2. Prove $V \leq \Phi$.

For a given state vector $(W_t, X_t, B_t) = (W, X, B)$ at time t and an arbitrary increasing trading time sequence $\{\tau_i\}_{i=1}^{\infty}$ where $t \leq \tau_1 \leq \tau_2 \leq \cdots$, applying Itô's formula to a sufficiently smooth function $\Phi(W, X, B)$, we obtain:

$$\begin{aligned} & \Phi(W, X, B) \\ = \ & \mathbb{E}\Big[e^{-\delta(\tau_1 \wedge \tau_L - t)}\Phi(W_{\tau_1\wedge\tau_L}, X_{\tau_1\wedge\tau_L}, B_{\tau_1\wedge\tau_L}) - \int_t^{\tau_1\wedge\tau_L} e^{-\delta(s-t)}\mathcal{L}\Phi(W_s, X_s, B_s)ds\Big] \ . \end{aligned}$$

Since Φ given in (IC.4) is a smooth supersolution of the variational inequality (A.1), we have the following inequality:

$$\begin{aligned} & \Phi(W, X, B) \\ \geq \ & \mathbb{E}\Big[e^{-\delta(\tau_1 - t)}F_\Phi(W_{\tau_1}, X_{\tau_1}, B_{\tau_1})\mathbf{1}_{\{\tau_2<\tau_L\}} + e^{-\delta(\tau_L - t)}U(G_{\tau_L}, B_{\tau_L})\Big] \\ = \ & \mathbb{E}\Big[e^{-\delta(\tau_1 - t)}\Big(U(G_{\tau_1}, B_{\tau_1}) + \Phi(W_{\tau_1+}, X_{\tau_1+}, B_{\tau_1+})\Big)\mathbf{1}_{\{\tau_2<\tau_L\}} + e^{-\delta(\tau_L - t)}U(G_{\tau_L}, B_{\tau_L})\Big], \end{aligned}$$

where

$$X_{\tau_1+} = \underset{\widehat{X}\geq -\widehat{W}/\kappa}{\arg\max} \ \Phi(\widehat{W}, \widehat{X}, \widehat{X})$$

and $\widehat{W} = W_{\tau_1} + (1-\theta_s)X_{\tau_1} - (1+\theta_p)\widehat{X}$. Using induction, we obtain for $n \geq 1$,

$$\begin{aligned} \Phi(W, X, B) \ \geq \ & \mathbb{E}\Big[\Big(\sum_{i=1}^{n} e^{-\delta(\tau_i - t)}U(G_{\tau_i}, B_{\tau_i}) + e^{-\delta(\tau_n - t)}\Phi(W_{\tau_n+}, X_{\tau_n+}, B_{\tau_n+})\Big)\mathbf{1}_{\{\tau_n<\tau_L\}} \\ & + e^{-\delta(\tau_L - t)}U(G_{\tau_L}, B_{\tau_L})\Big] \ . \end{aligned}$$

The proposed conditions (A.7)-(A.9) imply that $e^{-\delta(\tau_n - t)}\Phi(W_{\tau_n+}, X_{\tau_n+}, B_{\tau_n+}) \to 0$ almost surely, as $n \to \infty$. Therefore, we obtain

$$\Phi(W, X, B) \geq \mathbb{E}\Big[\sum_{i=1}^{\infty} e^{-\delta(\tau_i - t)}U(G_{\tau_i}, B_{\tau_i})\mathbf{1}_{\{\tau_i<\tau_L\}} + e^{-\delta(\tau_L - t)}U(G_{\tau_L}, B_{\tau_L})\Big].$$

We conclude that $\Phi(W, X, B) \geq V(W, X, B)$ immediately follows as $\{\tau_i\}_{i=1}^{\infty}$ is arbitrarily chosen and V in (14) is the value function associated with optimal policies.

Sketch of Verification of Transversality Conditions for Jump-Diffusion Models. Similar to our verification of transversality conditions for diffusion models, it is sufficient to show that $\phi(w,x)$ defined in (IC.5) satisfies:

$$\max\left\{\mathscr{L}^{\mathcal{J}}\phi(w,x),\ f_\phi(w,x)-\phi(w,x)\right\}\le 0, \tag{IC.10}$$

where $f_\phi(w,x)$ is given by (IC.7) and the differential operator $\mathscr{L}^{\mathcal{J}}$ is defined as:

$$\mathscr{L}^{\mathcal{J}}\phi = \frac{\sigma^2}{2}x^2\phi_{xx}+(\mu-r)x\phi_x-(\delta-\beta r)\phi+\rho\,\mathbb{E}[\phi(Yx)-\phi(x)]. \tag{IC.11}$$

We verify (IC.10) as follows. First, using the same argument as we did for the diffusion model, we can show $f_\phi(w,x)\le\phi(w,x)$. Second, substituting $\phi(w,x)$ given by (IC.5) into (IC.11) and using $(1+x)^\beta\le 1+\beta x$ (for $0\le\beta\le 1$ and $x\ge 0$), we obtain:

$$\mathscr{L}^{\mathcal{J}}\phi(w,x)\le h^{\mathcal{J}}(\alpha_+)\,C_1x^{\alpha_+}+\widehat{h}^{\mathcal{J}}(z)\,C_2[w+(1-\theta_s)x]^\beta\,, \tag{IC.12}$$

where

$$h^{\mathcal{J}}(\alpha_+) = \frac{1}{2}\sigma^2\alpha_+(\alpha_+-1)+(\mu-r)\alpha_+-(\delta-\beta r)+\rho\,\mathbb{E}[Y^\eta-1], \tag{IC.13}$$

$$\widehat{h}^{\mathcal{J}}(z) = \frac{1}{2}\sigma^2\beta(\beta-1)z^2+(\mu-r+\rho\,\mathbb{E}[Y])\beta z-(\delta-\beta r)\,, \tag{IC.14}$$

and

$$z=\frac{(1-\theta_s)x}{w+(1-\theta_s)x}\in[0,1]\,. \tag{IC.15}$$

Then, the proposed transversality conditions (IC.1)-(IC.3) imply that $\mathscr{L}^{\mathcal{J}}\phi(w,x)\le 0$.

ID Existence and Uniqueness of Variational Inequality (A.1) and Verification Theorem

In this appendix, we first prove the existence and uniqueness of the variational inequality (A.1), and then offer a verification theorem. To ease exposition and with no loss of generality, we focus on the case with no liquidity shocks: $\xi=0$.

Proposition ID.1. *For the $\xi=0$ case, the value function $V(W,X,B)$ of the optimization problem* (14) *is a unique viscosity solution to the variational inequality* (A.1).

For the existence proof, we use the method based on a fixed-point-theorem argument inspired by He and Yang (2019).[47] For the uniqueness proof, we use a comparison principle as in Øksendal and Sulem (2002) and Altarovici, Reppen, and Soner (2017).

Next, we introduce a related optimal-stopping problem to aid our existence proof.

[47] We can also prove the existence result by directly studying the original impulse-control problem (14).

A Related Optimal-stopping Problem. Consider the following optimal-stopping problem. For each $a \in \mathbb{R}$, let

$$v(w, x; a) := \sup_{\tau \geq t} \mathbb{E}_t \Big[e^{-\delta_e(\tau - t)} f(w_{\tau-}, x_{\tau-}; a) \Big] \,, \tag{ID.1}$$

where $\delta_e = \delta - \beta r$, the payoff function $f(w, x; a)$ is given by

$$f(w, x; a) := u\Big((1 - \theta_s)x - 1\Big) + (w + (1 - \theta_s)x)^\beta a \ , \tag{ID.2}$$

$dw_t = 0$, and

$$dx_t = (\mu - r)x_t dt + \sigma x_t d\mathcal{Z}_t \,.$$

Then, the value function $v(w, x; a)$ is a unique viscosity solution of the following variational inequality (see e.g., chapter 5 in Pham (2009)):

$$\max\left\{\mathscr{L}v(w, x; a),\ f(w, x; a) - v(w, x; a)\right\} = 0 \quad \text{for} \quad (w, x) \in \mathcal{S} \,. \tag{ID.3}$$

where

$$\mathscr{L}v = \frac{1}{2}\sigma^2 x^2 v_{xx} + (\mu - r)x v_x - \delta_e v \ .$$

Now, we can equivalently map the solution to our investor's problem analyzed in Section 2, which is characterized by the variational inequality (A.4), to the fixed point $v(w, x; a)$ satisfying (ID.1), where a is chosen as follows:

$$a = \max_{w \geq -\kappa} \frac{v(w, 1; a)}{(w + 1 + \theta_p)^\beta} < \infty \ . \tag{ID.4}$$

The following lemma shows that the object a given in (ID.4) is well defined.

Lemma ID.1. *For $a \in \mathbb{R}$, $v(w, 1; a)$ is continuous and satisfies the growth condition:*

$$v(w, 1; a) \leq C\Big[1 + (w + 1 - \theta_s)^\beta \, |a|\Big] \ , \tag{ID.5}$$

where C is a positive constant, and

$$\max_{w \geq -\kappa} \frac{v(w, 1; a)}{(w + 1 + \theta_p)^\beta} < \infty \ .$$

Proof. Since $f(w, x; a)$ is continuous and monotonic in w, the continuity of $v(w, 1; a)$ follows from the monotone convergence theorem. Next, the inequality (ID.5) follows from the result that $v(w, x; a) \leq C(x^{\alpha+} + [w + (1 - \theta_s)x]^\beta |a|)$, which we can prove using an argument similar to the one we used when verifying the transversality conditions. □

Lemma ID.2. *Let $v(w, x; a_i)$ be the value function defined in* (ID.1) *with $a = a_i$ for $i = 1, 2$ where $a_1 > a_2$. Then, the following inequalities hold:*

$$0 \leq v(w, x; a_1) - v(w, x; a_2) \leq [w + (1 - \theta_s)x]^{\beta}(a_1 - a_2) \ . \tag{ID.6}$$

Proof. First, we prove $v(w, x; a_2) \leq v(w, x; a_1)$. Note that

$$\begin{aligned} f(w, x; a_2) - v(w, x; a_1) &= u((1 - \theta_s)x - 1) + [w + (1 - \theta_s)x]^{\beta} a_2 - v(w, x; a_1) \\ &= f(w, x; a_1) - v(w, x; a_1) + [w + (1 - \theta_s)x]^{\beta}(a_2 - a_1) \leq 0, \end{aligned}$$

where the inequality follows from $a_2 < a_1$. Second, the variation inequality implies that $\mathscr{L} v(w, x; a_1) \leq 0$. Combining these two inequalities, we obtain

$$\max\{\mathscr{L} v(w, x; a_1),\ f(w, x, a_2) - v(w, x; a_1)\} \leq 0 \ ,$$

which implies that $v(w, x; a_2) \leq v(w, x; a_1)$ via the comparison principle (Pham, 2009).

Next, we show $v(w, x; a_1) \leq v(w, x; a_2) + [w + (1 - \theta_s)x]^{\beta}(a_1 - a_2)$. This inequality follows from

$$\begin{aligned} & f(w, x; a_1) - v(w, x; a_2) - [w + (1 - \theta_s)x]^{\beta}(a_1 - a_2) \\ = \ & u((1 - \theta_s)x - 1) + [w + (1 - \theta_s)x]^{\beta} a_2 - v(w, x; a_2) \\ = \ & f(w, x; a_2) - v(w, x; a_2) \leq 0, \end{aligned}$$

as $v(w, x; a_2)$ solves (ID.3).

Using the variational inequality, $\mathscr{L} v(w, x; a_2) \leq 0$, we have

$$\begin{aligned} & \mathscr{L}\{v(w, x; a_2) + [w + (1 - \theta_s)x]^{\beta}(a_1 - a_2)\} \\ = \ & \mathscr{L} v(w, x; a_2) + \mathscr{L}\{[w + (1 - \theta_s)x]^{\beta}(a_1 - a_2)\} \\ \leq \ & (a_1 - a_2)\mathscr{L}\{[w + (1 - \theta_s)x]^{\beta}\} \\ = \ & (a_1 - a_2)[w + (1 - \theta_s)x]^{\beta}\, \widehat{h}\,(z)\,, \end{aligned}$$

where

$$\widehat{h}(z) = \frac{1}{2}\sigma^2\beta(\beta - 1)z^2 + (\mu - r)\beta z - \delta_e \tag{ID.7}$$

and

$$z = \frac{(1 - \theta_s)x}{w + (1 - \theta_s)x} \in \left[0, \frac{1 - \theta_s}{1 - \theta_s - \kappa}\right] \ . \tag{ID.8}$$

Equation (ID.8) uses the leverage constraint $w \geq -\kappa x$.

Using the transversality conditions, we can show that $\widehat{h}(z) \leq 0$. Therefore, $v(w, x; a_2) + [w + (1 - \theta_s)x]^{\beta}(a_1 - a_2)$ is a supersolution to the variational inequality (ID.3) with $a = a_1$. Finally, using the comparison principle, we obtain the following inequality: $v(w, x; a_1) \leq v(w, x; a_2) + [w + (1 - \theta_s)x]^{\beta}(a_1 - a_2)$. □

Existence. Next, we prove the existence of a solution to the variational inequality (A.4), using a fixed-point argument inspired by the proof of Theorem 3.6 in He and Yang (2019).

Proof of Proposition ID.1 (existence). First, we introduce a linear space $\mathbf{X}$ of continuous functions on $[-\kappa, \infty)$ with a finite norm $\|\cdot\|$ defined as:

$$\|\zeta\| := \max_{w\in[-\kappa,\infty)} \frac{|\zeta(w)|}{(w+1+\theta_p)^{\beta}}\,, \quad \forall\, \zeta \in \mathbf{X}\,.$$

We can show that $(\mathbf{X}, \|\cdot\|)$ is a Banach space.

Second, consider the mapping on $(\mathbf{X}, \|\cdot\|)$: for each $\zeta \in \mathbf{X}$, define $\mathcal{F}(\zeta) = v(w, 1; a)$ for $w \in [-\kappa, \infty)$, where $v(w, x; a)$ is the solution to the variational inequality (ID.3) with $a = \|\zeta\|$. Then, Lemma ID.1 implies $\mathcal{F}(\zeta) \in \mathbf{X}$.

Third, from Lemma ID.2, we conclude that[48]

$$\begin{aligned}\|\mathcal{F}(\zeta_1) - \mathcal{F}(\zeta_2)\| &= \max_{w\in[-\kappa,\infty)} \frac{\mathcal{F}(\zeta_1) - \mathcal{F}(\zeta_2)}{(w+1+\theta_p)^{\beta}} \\ &\le \max_{w\in[-\kappa,\infty)} \frac{(w+1-\theta_s)^{\beta}}{(w+1+\theta_p)^{\beta}}(\|\zeta_1\| - \|\zeta_2\|) \le \|\zeta_1 - \zeta_2\|.\end{aligned}$$

By Schauder fixed-point theorem, there exists at least one fixed point $\zeta^* \in \mathbf{X}$ such that $\mathcal{F}(\zeta^*) = \zeta^*$. □

Uniqueness. The uniqueness result is implied by the following comparison principle.

Proposition ID.2 (Comparison principle)**.** *Suppose that V_1 and V_2 are respectively classical super-solution and sub-solution to the variational inequality* (A.1)*:*

$$\max\{\mathcal{L}V_1,\ F_1 - V_1\} \le 0 \quad \textit{and} \quad \max\{\mathcal{L}V_2,\ F_2 - V_2\} \ge 0, \tag{ID.9}$$

where

$$\begin{aligned}\mathcal{L}V_i &= \frac{1}{2}\sigma^2 X^2 \partial_{XX} V_i + \mu X \partial_X V_i + rW\partial_W V_i + rB\partial_B V_i - \delta V_i\,,\\ F_i &= U\big((1-\theta_s)X - B, B\big) + \max_{\widehat{X}}\ V_i\big(\widehat{W}, \widehat{X}, \widehat{X}\big)\,.\end{aligned}$$

and $\widehat{W} = W + (1-\theta_s)X - (1+\theta_p)\widehat{X}$. Moreover, V_1 and V_2 satisfy the following growth conditions:

$$V_1(W,X,B) \le C\,B^{\beta}\Big[\Big(\frac{X}{B}\Big)^{\alpha_+} + \Big(\frac{W}{B} + (1-\theta_s)\frac{X}{B}\Big)^{\beta}\Big], \tag{ID.10}$$

$$V_2(W,X,B) \ge -C\,B^{\beta}\Big[\Big(\frac{X}{B}\Big)^{\alpha_+} + \Big(\frac{W}{B} + (1-\theta_s)\frac{X}{B}\Big)^{\beta}\Big], \tag{ID.11}$$

for a constant $C > 0$. Then, we have $V_1 \ge V_2$.

[48] Because of w, $\mathcal{F}$ is not a contraction mapping. Therefore, different from He and Yang (2019), the uniqueness result in our model cannot be obtained from the contraction mapping theorem.

Proof. Step 1. Construct a strict super solution to the variational inequality (A.1). Given $\varepsilon > 0$ and $\gamma \in [\alpha^+, 1]$, define

$$V_\varepsilon(W,X,B) := V_1(W,X,B) + \varepsilon\zeta(W,X,B) \quad \text{with } \zeta(W,X,B) = (W+X)^\gamma \ .$$

For each compact set Ω in the solvency region,

$$\begin{aligned} & U\big((1-\theta_s)X - B, B\big) + \max_{\widehat{X}} \ V_\varepsilon\big(\widehat{W}, \widehat{X}, \widehat{X}\big) - V_\varepsilon(W,X,B) \\ \leq \ & U\big((1-\theta_s)X - B, B\big) + \max_{\widehat{X}} \ V_1\big(\widehat{W}, \widehat{X}, \widehat{X}\big) + \varepsilon \max_{\widehat{X}} \ \zeta\big(\widehat{W}, \widehat{X}, \widehat{X}\big) - V_\varepsilon(W,X,B) \\ = \ & F_1(W,X,B) - V_1(W,X,B) + \varepsilon\Big\{ \max_{\widehat{X}} \ \zeta\big(\widehat{W}, \widehat{X}, \widehat{X}\big) - \zeta(W,X,B)\Big\} \\ \leq \ & \varepsilon\Big\{ \max_{\widehat{X}} \Big(W + (1-\theta_s)X - \theta_p\widehat{X})\Big)^\gamma - (W+X)^\gamma\Big\} \\ \leq \ & \varepsilon(W+X)^\gamma\Big[\Big(1 - \frac{\theta_s X}{W+X}\Big)^\gamma - 1\Big] \leq -\varepsilon C, \end{aligned}$$

where $C > 0$ is a constant that depends on Ω.

For each compact set Ω in the solvency region, provided that δ is sufficiently large,

$$\begin{aligned} \mathcal{L}V_\varepsilon(W,X,B) \ &\leq \ \varepsilon\mathcal{L}\zeta = \varepsilon\Big\{\frac{1}{2}\sigma^2X^2\zeta_{XX} + \mu X\zeta_X + rW\zeta_W + rB\zeta_B - \delta\zeta\Big\} \\ &= \ \varepsilon(W+X)^\gamma\Big\{\frac{1}{2}\sigma^2\gamma(\gamma-1)\Big(\frac{X}{W+X}\Big)^2 + (\mu - r)\gamma\frac{X}{W+X} + r\gamma - \delta\Big\} \\ &\leq \ -\varepsilon C, \end{aligned}$$

where $C > 0$ depends on Ω.

Therefore, on each compact set Ω, V_ε is a strict super solution, which implies

$$\max\{\mathcal{L}V_\varepsilon, \ F_\varepsilon - V_\varepsilon\} \leq -\varepsilon C \quad \text{in } \ \Omega \ ,$$

where

$$F_\varepsilon = U\big((1-\theta_s)X - B, B\big) + \max_{\widehat{X}} \ V_\varepsilon\big(\widehat{W}, \widehat{X}, \widehat{X}\big) \ .$$

Step 2. Next, we provide by contradiction. Suppose that

$$\max\{V_2(W,X,B) - V_1(W,X,B)\} > 0.$$

Then, choose ε such that

$$m := \max\{V_2(W,X,B) - V_\varepsilon(W,X,B)\} > 0.$$

The growth conditions (ID.10)-(ID.11) imply that the maximizer is in a compact set. At this maximizer, $(V_2 - V_\varepsilon)_{XX} \leq 0$ and $(V_2 - V_\varepsilon)_Y = 0$ for $Y \in \{W, X, B\}$. Then we have

$$\mathcal{L}(V_2 - V_\varepsilon) \leq -\delta(V_2 - V_\varepsilon) < 0,$$

which implies $\mathcal{L}V_2(W_0, X_0, B_0) < \mathcal{L}V_\varepsilon(W_0, X_0, B_0) < 0$. Since V_2 is a subsolution, we must have $F_2 - V_2 \geq 0$. Recall $F_\varepsilon - V_\varepsilon \leq -\varepsilon C$ from Step 1. Therefore, at the maximizer point.

$$
\begin{aligned}
m &= V_2 - V_\varepsilon \leq F_2 - F_\varepsilon - \varepsilon C \\
&= \max_{\widehat{X}} V_2\big(\widehat{W}, \widehat{X}, \widehat{X}\big) - \max_{\widehat{X}} V_\varepsilon\big(\widehat{W}, \widehat{X}, \widehat{X}\big) - \varepsilon C \\
&\leq \max_{\widehat{X}} \{V_2\big(\widehat{W}, \widehat{X}, \widehat{X}\big) - V_\varepsilon\big(\widehat{W}, \widehat{X}, \widehat{X}\big)\} - \varepsilon C \\
&\leq m - \varepsilon C < m,
\end{aligned}
$$

which is a contradiction. □

We can adapt the above proof to the case where sub- and super-solutions are in the viscosity sense (see, e.g., the proof of Theorem 3.8 in Øksendal and Sulem (2002)).

Next we provide a verification theorem.

Proposition ID.3 (Verification)**.** *Let $v(w, x)$ be a solution to* (A.4) *with a boundary condition $v(w, x) = f(w, x)$ at $x = -w/\kappa > 0$, satisfying certain regularity conditions. And let $f(w, x)$ and $\widehat{v}$ be defined by* (23) *and* (24), *respectively. Then, $v(w, x)$ is the value function of the optimization problem* (22) *and solves the HJB equation* (26) *in the holding domain $\mathcal{H}$ where $v(w, x) > f(w, x)$. The optimal polices are given as follows.*

1. *Starting from $(w, x) \in \mathcal{H}$ at time t, the next optimal realization time is the first passage time of the process (w_t, x_t): $\tau = \inf\{s \geq t \,|\, (w_s, x_s) \notin \mathcal{H}\}$ subject to the dynamics* (20) *and* (21).

2. *For each future investing episode, the optimal allocation strategy is to set the starting position at $(w^*, 1) \in \mathcal{H}$, where $w^* = \arg\max_{w \geq -\kappa} m(w)$ is the optimal allocation ratio between the risk-free asset and the stock the investor chooses.*

Proof. Our proof is standard as in the optimal-stopping literature (see, e.g., Theorem 2.1 in Øksendal and Sulem (2002)). We thus only provide a sketch. With the regularity assumptions for $v(w, x)$, we apply Itô's formula to $e^{-\delta_e(s-t)}v(w_s, x_s)$ between t and any trading time $\tau \geq t$. Using the exponential distribution of τ_L and the variational inequality

(A.4), we obtain

$$
\begin{aligned}
v(w,x) &= \mathbb{E}_t\Big[e^{-\delta_e(\min(\tau,\tau_L)-t)}v(w_{\min(\tau,\tau_L)},x_{\min(\tau,\tau_L)}) \\
&\quad - \int_t^{\min(\tau,\tau_L)} e^{-\delta_e(s-t)}\Big(\frac{1}{2}\sigma^2x^2v_{xx}(w_s,x_s)+(\mu-r)xv_x(w_s,x_s)-\delta_e v(w_s,x_s)\Big)ds\Big] \\
&= \mathbb{E}_t\Big[e^{-\delta_e(\tau-t)}v(w_\tau,x_\tau)\mathbf{1}_{\{\tau<\tau_L\}}+e^{-\delta_e(\tau_L-t)}v(w_{\tau_L},x_{\tau_L})\mathbf{1}_{\{\tau\geq\tau_L\}} \\
&\quad - \int_t^{\tau} e^{-\delta_e(s-t)}\mathscr{L}v(w_s,x_s)ds\Big] \\
&\geq \mathbb{E}_t\Big[e^{-\delta_e(\tau-t)}f(w_\tau,x_\tau)\mathbf{1}_{\{\tau<\tau_L\}}+e^{-\delta_e(\tau_L-t)}u((1-\theta_s)x_{\tau_L}-1)\Big]
\end{aligned}
$$

Therefore, as τ can be arbitrarily chosen, $v(w,x)$ cannot be lower than the value function. Finally, the equality result for the last inequality follows from the strategies proposed in the proposition. □