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Managing government debt

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To construct a stochastic version of [R. J. Barro, *J. Polit. Econ.* 87, 940–971 (1979)] normative model of tax rates and debt/GDP dynamics, we add risks and markets for trading them along lines suggested by [K. J. Arrow, *Rev. Econ. Stud.* 31, 91–96 (1964)] and [R. J. Shiller, Creating Institutions for Managing Society's Largest Economic Risks (OUP, Oxford, 1994)]. These modifications preserve Barro's prescriptions that a government should keep its debt-gross domestic product (GDP) ratio and tax rate constant over time and also prescribe that the government insure its primary surplus risk by selling or buying the same number of shares of a Shiller macro security each period.

tax smoothing | Ricardian equivalence | risk premium

Significance

A government that starts out either owing or owning debt should trade risky Shiller GDP securities that hedge its primary surplus risk. The indirect utility functional from a government debt and tax control problem that maximizes the expected utility of a representative consumer is a government loss function that appears in widely used models of tax smoothing.

We construct a normative model of taxation and government debt management that adds risks and opportunities to Barro (1)'s nonstochastic model of optimal tax-debt policies. We begin by deducing Barro's government loss function as the indirect utility functional that emerges from maximizing the expected utility of a representative consumer. Having added risk to Barro's environment, we also add markets in one-period ahead Arrow (2) securities that allow decision makers to insure those risks. Given the fiscal risks that it chooses to take, the government chooses to trade a single Shiller (3) security whose payoff is proportional to GDP growth.*

We adopt a setup close to one that Lucas (7, 8, section III) used to measure potential benefits from improving countercyclical macroeconomic policies. We incorporate insights of Hansen et al. (9), Alvarez and Jermann (10), and Barillas et al. (11) about the information about costs of business cycles that is contained in a stochastic discount factor (SDF) process. Like Lucas (7, 8, section III), an SDF process is determined outside our model. We specify the SDF process to be a simple generalization of the time-discount factor process assumed by Barro (1).

Our model of optimal taxation and debt management preserves the constant tax rate and constant debt-GDP ratio prescriptions of Barro (1), and adds the prescription that the government sells shares in a Shiller (3) claim to GDP. Our government uses a formula of Jiang et al. (12, 13) for discounting risk-free government debt. It adds a risk premium to the formula used by Blanchard (14), the ultimate source of which is shortfalls in the government's risky primary government surplus stream as a sole source of backing for its risk-free debt.

1. The Setting

GDP $\{Y_t\}$ follows

$$Y_{t+1} = \exp\left(g - \frac{\sigma^2}{2} + \sigma\varepsilon_{t+1}\right) Y_t, \quad [1]$$

where $\varepsilon_{t+1} \sim \mathcal{N}(0,1)$ is an i.i.d. process. Government expenditures $\Gamma_t = \gamma Y_t$ are perpetually proportional to Y_t. Government debt B_0 is due at time 0. Total tax collections $\mathcal{T}_t$ at time $t \geq 1$ are a measurable function of the history $\varepsilon^t = [\varepsilon_t, \varepsilon_{t-1}, \cdots, \varepsilon_0]$; $\mathcal{T}_0$ is an initial condition chosen by the government. A stochastic discount factor process $\{M_t\}$ is determined outside our model and has multiplicative increments†

$$\frac{M_{t+1}}{M_t} \equiv m_{t+1} = \exp\left[-\left(r + \frac{\eta^2}{2}\right) - \eta\varepsilon_{t+1}\right], \quad [2]$$

where η is the price of GDP growth risk ε_{t+1} and $M_0 > 0$ is given. To price Arrow (2) securities, we can interpret the multiplicative increment $m(\varepsilon_{t+1})$ of the SDF process

*This is a consequence of a "dynamic trading" argument in the spirit of Harrison and Kreps (4), Black and Scholes (5), and Merton (6).

†This is a counterpart to and generalization of an assumption of Barro (1), who took a time-invariant risk-free interest rate as given.

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as an exogenous time t state-price density of a claim bought at time t that pays off at time $t+1$.[‡] To understand the sense in which $m(\varepsilon)$ is a density, where $\Phi_t(\cdot)$ is the standardized normal cumulative distribution function, an Arrow security that pays off 1 unit of GDP at time $t+1$ whenever the realized GDP shock lies in interval $(\varepsilon_{t+1}, \varepsilon_{t+1} + d\varepsilon_{t+1})$ is priced at time t by $m(\varepsilon_{t+1})d\Phi(\varepsilon_{t+1})$. Let a function $\Pi(\cdot) : (-\infty, \infty) \to \mathbf{R}$ represent a bundle of ε_{t+1}-contingent payoffs. The price at time t of bundle Π is $\int \Pi(\varepsilon)m(\varepsilon)d\Phi(\varepsilon)$.[§]

It is useful to view claims on GDP as a security and to use the SDF process with increments defined by Eq. 2 to price this Shiller (3) macro security:

$$S_t = \mathbb{E}_t\left[\sum_{u=t+1}^{\infty} \frac{M_u}{M_t} Y_u\right] = \frac{e^{-\delta} Y_t}{1 - e^{-\delta}}, \qquad [3]$$

where $\lambda = \eta\sigma$ and $\delta = r + \lambda - g$. The one-period gross return on this Shiller security is

$$R_{t+1} \equiv \frac{S_{t+1} + Y_{t+1}}{S_t} = \exp\left(r + \lambda - \frac{\sigma^2}{2} + \sigma\varepsilon_{t+1}\right), \qquad [4]$$

with expected return

$$\mathbb{E}_t[R_{t+1}] = e^{r+\lambda}.$$

Since the SDF process given in Eq. **2** implies that the price of a one-period risk-free bond is $\mathbb{E}_t(m_{t+1}) = e^{-r}$, $\lambda = \eta\sigma$ is the risk premium component of the continuously compounded return $(r+\lambda)$ on the Shiller security; it equals the price η of risk ε_{t+1} times the Shiller security's exposure to that risk: σ. To indicate the dependence of R_{t+1} on ε_{t+1}, we'll often write $R_{t+1} = R(\varepsilon_{t+1})$.

We assume that the government instead manages risks in a way recommended by Shiller (3). Appendix A shows that trading the Shiller security lets the government attain the same optimal outcomes as it can by trading Arrow securities.[¶]

2. Optimal Fiscal Policy

We provide a theory of how the government chooses stochastic processes for its tax and portfolio management policy $\{\tau_t, \Delta_t\}_{t=0}^{\infty}$. We follow Barro (1) and assume that raising revenues $\mathcal{T}_t$ brings distortions measured by $\Theta(\mathcal{T}_t, Y_t)$, where

$$\Theta(\mathcal{T}_t, Y_t) = \theta(\tau_t)Y_t \qquad [5]$$

and the scaled deadweight loss function $\theta(\tau)$ is increasing, convex, and smooth. The positive derivative $\theta'(\cdot)$ plays a key role in inducing the government to make total tax collections $\mathcal{T}_t$ be homogeneous of degree one in GDP so that primary surplus $\mathcal{T}_t - G_t$ also becomes homogeneous of degree one in GDP and subject to the same risk ε_{t+1} that affects GDP growth. If it wants to issue risk-free bonds, the government must insure that risk. A way to do that would be to purchase or sell an appropriate package of one-period Arrow (2) securities. The Shiller security is that appropriate package because the government faces primary surplus risk that is perfectly correlated with risk in the price of the Shiller security at time $t+1$.[#]

At time t the government purchases Δ_t shares of the Shiller security. Consequently, starting with an initial risk-free debt balance B_0, risk-free government debt $\{B_t\}$ evolves as

$$B_{t+1} = e^r B_t + e^r(\gamma - \tau_t)Y_t - \Delta_t\left(R_{t+1} - e^r\right)S_t, \quad t \geq 0, \qquad [6]$$

where $(\gamma - \tau_t)Y_t$ is the government's primary deficit at time t and $-\Delta_t\left(R_{t+1} - e^r\right)S_t$ describes how the government's purchase of Δ_t shares of the Shiller security at time t affects B_{t+1}.

Let C_t be time-t consumption of a representative consumer and let a one-period felicity function be $U(C) = \frac{C^{1-\psi}}{1-\psi}$, where $\psi > 0$ is a coefficient of relative risk aversion.[‖] A benevolent government wants a tax-portfolio policy $\{\tau_t, \Delta_t\}_{t=0}^{\infty}$ that maximizes an indirect utility function F_0 of a representative household defined by

$$F_0 = \max_{\{C_t\}} \mathbb{E}_0\left[\sum_{t=0}^{\infty} e^{-\rho t} U(C_t)\right]. \qquad [7]$$

The household's maximization is subject to the intertemporal budget constraint

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty} M_t C_t\right] \leq \quad W_0 + \mathbb{E}_0\left[\sum_{t=0}^{\infty} M_t\left(Y_t - \mathcal{T}_t - \Theta_t\right)\right], \qquad [8]$$

where W_0 is the household's initial exogenously endowed wealth that is unaffected by forces active in our model. The household takes SDF $\{M_t\}$, tax collection $\{\mathcal{T}_t\}$, and deadweight loss $\{\Theta_t\}$ processes as given. By modifying a Lagrangian method used by Cox and Huang (15) to allow for taxes and deadweight losses, we can deduce

$$F_0 = \frac{(\beta(W_0 + X_0))^{1-\psi}}{1-\psi}, \qquad [9]$$

where $\beta = \left[1 - \exp\left(-\left(\left(1 - \frac{1}{\psi}\right)r + \frac{\rho}{\psi} + \frac{1}{2}\left(1 - \frac{1}{\psi}\right)\frac{1}{\psi}\eta^2\right)\right)\right]^{-\frac{\psi}{1-\psi}}$ and

$$X_0 = \mathbb{E}_0\left[\sum_{s=0}^{\infty} \frac{M_s}{M_0}\left(Y_s - \mathcal{T}_s - \Theta_s\right)\right]. \qquad [10]$$

To maximize the household's utility function F_0 given in Eq. **9** over a taxation-portfolio policy $\{\tau_t, \Delta_t\}_{t=0}^{\infty}$, it suffices for the the government to choose a joint $\{\tau_t, \Delta_t\}_{t=0}^{\infty}$ process to maximize the present value of the households' payoffs $Y_s - \mathcal{T}_s - \Theta_s$ given in Eq. **10** subject to tax distortions Eq. **5**, initial condition (B_0, Y_0) and the government budget constraint

$$B_0 \leq \mathbb{E}_0\left[\sum_{u=0}^{\infty} \frac{M_u}{M_0}\left(\mathcal{T}_u - \Gamma_u\right)\right]. \qquad [11]$$

Evidently, maximizing the households' value given in Eq. **10** is equivalent to minimizing the present value of taxes $\mathcal{T}_t$ plus deadweight losses Θ_t; and both are equivalent to maximizing the

[‡]We abuse notation by using ε_{t+1} to represent both GDP shock and its realization.

[§]Consider a claim whose payoff is 1 for all ε_{t+1} at $t+1$ so that $\Pi(\varepsilon_{t+1}) = 1$. The time-$t$ price of this claim equals $\int \Pi(\varepsilon)m(\varepsilon)d\Phi(\varepsilon) = \int m(\varepsilon)d\Phi(\varepsilon) = e^{-r}$.

[¶]That is, as long as the country can use the Shiller security to manage risk, there is no need to have a complete set of Arrow securities at each date t.

[#]Appendix A.

[‖]The key result that the household's utility maximization requires that the maximization of the market value of income flows holds for any well-behaved increasing and concave utility function.

representative consumer's expected utility functional F_0 defined in Eq. **7**.

Before approaching this problem, we state:

Proposition 1. *To recover findings of Barro (16), suppose that the $\tau(\cdot)$ function satisfies $\theta'(\cdot) = 0$ for all tax rates τ. Then, any taxation-portfolio strategy $\{\tau_t, \Delta_t\}_{t=0}^{\infty}$ that satisfies the government's budget constraint solves the government's problem.*

Proposition 1 is Barro (16)'s Ricardian equivalence theorem. To make an optimal taxation-portfolio strategy profile determinant, Barro (1) injected tax distortion function $\theta(\cdot)$ with $\theta'(\cdot) > 0$.

Now turning to the government's problem in the presence of an increasing, convex, and smooth $\theta(\cdot)$ function, we formulate Eq. **10** as a dynamic programming problem in which $X_0^* = P(B_0, Y_0)$ is the maximal attainable X_0 that satisfies Eq. **10**. Value function $P(B_0, Y_0)$ satisfies the Bellman equation:

$$P(B_t, Y_t) = \max_{\tau_t, \Delta_t} \mathbb{E}_t\left[Y_t - \tau_t Y_t - \theta(\tau_t) Y_t + m_{t+1} P(B_{t+1}, Y_{t+1})\right]. \quad [12]$$

Substituting Eq. **6** into Eq. **12** gives

$$\begin{aligned} P(B_t, Y_t) = \max_{\tau_t, \Delta_t}\ & Y_t - \tau_t Y_t - \theta(\tau_t) Y_t \\ & + \mathbb{E}_t\left[m_{t+1} P\left(e^r B_t + e^r(\gamma - \tau_t) Y_t\right.\right. \\ & \left.\left. -\Delta_t\left(R_{t+1} - e^r\right) S_t, Y_{t+1}\right)\right]. \end{aligned} \quad [13]$$

First-order necessary conditions for τ_t and Δ_t, respectively,** are:

$$1 + \theta'(\tau_t) = -\mathbb{E}_t\left[m_{t+1} e^r P_B(B_{t+1}, Y_{t+1})\right], \quad [14]$$

and

$$-\mathbb{E}_t\left[m_{t+1}(R_{t+1} - e^r) P_B(B_{t+1}, Y_{t+1})\right] = 0. \quad [15]$$

Let $b_t = B_t / Y_t$ denote the debt-GDP ratio. We guess and verify that $P(B_t, Y_t) = p(b_t) Y_t$ and that $b_{t+1} - b_t = 0$ for all t. First-order condition Eq. **14** for tax rate τ_t implies

$$1 + c'(\tau_t) = -p'(b_t) = -p'(b_0), \quad [16]$$

which equates the marginal cost $1 + c'(\tau_t)$ of taxing the household with the marginal benefit $-p'(b_t)$ of reducing debt. Eq. **16** implies that the tax rate τ_t is constant over time. Combining this outcome with the government's budget constraint Eq. **6** implies

$$\tau_t = (1 - e^{-\delta}) b_t + \gamma. \quad [17]$$

The optimal tax rate is constant over time and independent of the deadweight cost function $\theta(\cdot)$. To verify $b_{t+1} - b_t = 0$ for all $t \geq 0$, it is necessary that

$$\frac{B_{t+1}}{Y_{t+1}} = \frac{e^r B_t + e^r(\gamma - \tau_t) Y_t - \Delta_t\left(R_{t+1} - e^r\right) S_t}{Y_{t+1}} = \frac{B_t}{Y_t}, \quad [18]$$

which implies:††

$$\Delta_t = -\left(1 - e^{-\delta}\right) b_t = -\left(1 - e^{-\delta}\right) b_0. \quad [19]$$

If $b_0 = 0$, then the government sets $\tau_t = \gamma$ and $b_t = b_0$ for all $t \geq 0$, there is no need to manage risk, holding of the Shiller security $\Delta_t = 0$ for all $t \geq 0$. If $b_0 > 0$, then for the government to honor its debt, $\tau_t - \gamma > 0$ for all $t \geq 0$. Because it permanently runs a primary surplus at a rate proportional to Y_t, its net liability B_{t+1} is risky. So the government lowers the exposure to risk that B_{t+1} presents by taking a short position in the Shiller security. Thus, it sets $\Delta_t < 0$ as prescribed by Eq. **19**. Doing that makes the net risk exposure of $(B_{t+1} - B_t)/B_t$ equal that of $(Y_{t+1} - Y_t)/Y_t$, enabling the government to sustain $b_t = b_0 > 0$ for all $t \geq 0$. Symmetrically, if $b_0 < 0$, the government permanently runs a primary deficit at a rate proportional to Y_t. That makes the stock of B_{t+1} less risky than the government wants, inducing it to take a long position in the Shiller security by setting $\Delta_t > 0$ as prescribed by Eq. **19**. Doing that sets the net risk exposure of $(B_{t+1} - B_t)/B_t$ equal to that of $(Y_{t+1} - Y_t)/Y_t$, which allows the government to sustain $b_t = b_0 < 0$ for all $t \geq 0$.

The random vector $[R_{t+1}, m_{t+1}]^\top$ is bivariate normal, so

$$\begin{aligned} \mathbb{E}_t\left[m_{t+1} R_{t+1}\right] &= e^{r + \lambda - \frac{\sigma^2}{2}} \mathbb{E}_t\left[m_{t+1} e^{\sigma \varepsilon_{t+1}}\right] \\ &= \mathbb{E}_t\left[e^r m_{t+1}\right]. \end{aligned} \quad [20]$$

Outcome Eq. **19** and $b_{t+1} - b_t = 0$ for all $t \geq 0$ confirm that the first-order condition Eq. **15** holds.

In summary, we have established:

Theorem 2. *The optimal fiscal plan is described by $b_t = b_0$ and the following three equations:*

1. *Optimal tax rate:*

$$\tau(b_t) = \tau(b_0) = (1 - e^{-\delta}) b_0 + \gamma. \quad [21]$$

2. *Optimal purchase of Shiller security:*

$$\Delta_t = \Delta_0 = -\left(1 - e^{-\delta}\right) b_0. \quad [22]$$

3. *The GDP-scaled value X_0/Y_0 of after-tax, after tax-distortions GDP flowing to households:*

$$p(b_t) = p(b_0) = \frac{1 - \tau(b_0) - \theta(\tau(b_0))}{1 - e^{-\delta}}. \quad [23]$$

Theorem 2 tells the government how to smooth taxes and to manage its debt.‡‡ Eqs. **21** to **23** can be solved recursively. Use

** The second-order condition with respect to τ_t holds because $\theta(\tau)$ is convex. The second-order condition with respect to Δ_t is

$$\mathbb{E}_t\left[m_{t+1}(R_{t+1} - e^r)^2 P_{BB}(B_{t+1}, Y_{t+1})\right] < 0$$

because $P_{BB} < 0$, as we verify later.

†† Substituting $\tau_t = (1 - e^{-\delta}) b_t + \gamma$ and Eq. **3** into Eq. **18**, we obtain:

$$\begin{aligned} \Delta_t &= \frac{e^r B_t - e^r(1 - e^{-\delta}) B_t}{(R_{t+1} - e^r) S_t} - \frac{B_t}{(R_{t+1} - e^r) S_t} \frac{Y_{t+1}}{Y_t} \\ &= \frac{(1 - e^{-\delta}) e^r B_t}{(R_{t+1} - e^r) Y_t} - \frac{(1 - e^{-\delta}) e^{\delta} B_t}{(R_{t+1} - e^r) Y_t} e^{\left(g - \frac{1}{2}\sigma^2\right) + \sigma \varepsilon_{t+1}} \\ &= -(1 - e^{-\delta}) \frac{B_t}{Y_t} \left(\frac{e^{\delta + \left(g - \frac{1}{2}\sigma^2\right) + \sigma \varepsilon_{t+1}} - e^r}{R_{t+1} - e^r}\right) \\ &= -(1 - e^{-\delta}) b_t. \end{aligned}$$

‡‡ To connect with findings of Barro (16) and Barro (1), note that when taxation brings no deadweight losses, i.e., when $\theta(\tau) = 0$, Ricardian equivalence holds because $p(b) = \frac{1-\gamma}{1-e^{-\delta}} - b$ and $p'(b) = -1$ for all b.

Eq. **21** to compute a tax rate $\tau(b_0)$ and then set $\tau_t = \tau(b_0)$ for all $t \geq 0$. This tax rate suffices to fund government expenditures and to service the government's risk-free debt, including costs that arise from selling the Shiller security in the amount recommended by Eq. **22**. The optimal Shiller security position Δ_t is negative and constant over time for $b_0 > 0$. Finally, $p(b_0)$ in Eq. **23** is the (scaled) value of revenues after taxes and after deadweight losses from taxation flowing to households: $p(b_0) = X_0/Y_0$, where X_0 is defined in Eq. **10** under the optimal policy.

3. Who Owns GDP?

By applying some asset pricing formulas, we can summarize how an optimal government policy distributes costs and benefits. The time-0 value of GDP is

$$V_0 = \mathbb{E}_0\left[\sum_{t=0}^{\infty} M_t Y_t\right] = \frac{Y_0}{1-e^{-\delta}}. \quad [24]$$

Our "value distribution formula" is:

$$V_0 = P(B_0, Y_0) + B_0 + [PV(\Gamma) + PV(\Theta)], \quad [25]$$

where

- the value of after-tax, after tax-distortions GDP flowing to households is

$$P(B_0, Y_0) = \mathbb{E}_0\left[\sum_{t=0}^{\infty} M_t \left(Y_t - \tau_t Y_t - \Theta(\mathcal{T}_t, Y_t)\right)\right] = \frac{(1-\tau_0-\theta(\tau_0))\,Y_0}{1-e^{-\delta}}. \quad [26]$$

- the value of risk-free government debt is

$$B_0 = \mathbb{E}_0\left[\sum_{t=0}^{\infty} M_t \left(\mathcal{T}_t - \Gamma_t\right)\right] = \frac{(\tau_0-\gamma)Y_0}{1-e^{-\delta}}. \quad [27]$$

- the value of government spending is

$$PV(\Gamma) = \mathbb{E}_0\left[\sum_{t=0}^{\infty} M_t \Gamma_t\right] = \frac{\gamma Y_0}{1-e^{-\delta}}. \quad [28]$$

- the value of deadweight taxation loss is

$$PV(\Theta) = \mathbb{E}_0\left[\sum_{t=0}^{\infty} M_t \Theta_t\right] = \frac{\theta(\tau_0) Y_0}{1-e^{-\delta}}. \quad [29]$$

Evidently, after government expenditures, taxes, and debt-servicing costs are taken into account, three shareholders own GDP: private households own $(1-\tau_0-\theta(\tau_0))$ shares; government creditors own share $(\tau_0-\gamma)$; and the government tax and spending authority, being a pass-through, owns and spends share $(\gamma+\theta(\tau_0))$. All three owners hold cum-dividend shares of the Shiller security. The three claimants have equal priority.

Remark 3. An increasing, convex, and smooth deadweight cost $\theta(\cdot)$ function induces the government to manage its exposure to GDP risk ε_{t+1} by keeping both the tax rate τ_t and the debt-GDP ratio b_t constant.[§§] Investors are willing to hold risk-free one-period debt at a gross interest rate e^r. The government's risky primary surplus process $\{(\tau_0-\gamma)Y_t; t \geq 0\}$ constitutes the ultimate "backing" behind all of its debts. To minimize deadweight costs of taxes, the government insures against primary surplus risk risk ε_{t+1} by selling $\left(1-e^{-\delta}\right) b_0 Y_t$ shares of the Shiller security each period. The government pays a "portfolio management cost" in the form of the Shiller security's risk premium λ per unit of time.

Remark 4. The government services B_t with the fiscal surplus stream $\mathcal{T}_t - \Gamma_t$ and by trading Δ_t shares of the Shiller security. The government could instead issue $(\tau_0-\gamma)$ shares of the (cum-dividend) Shiller security at time 0 and use the proceeds to retire B_0. By issuing shares in the Shiller security in that way, the government would in effect be selling a constant fraction of the country's perpetual stream of output that it commandeers by taxing; these shares of the Shiller security would be fully backed by the government's perpetual stream of fiscal surpluses. Consequently, government debt B_0 has the same value and risk as $(\tau_0-\gamma)$ shares of the (cum-dividend) Shiller security. Selling shares of the Shiller security in this way at time 0 would in effect be conducting a "debt-equity swap."

4. Concluding Remarks

By allowing the government to trade either a complete set of one-period Arrow securities or a single Shiller (3) security, we have extended Barro (1) model in a way that preserves salient prescriptions: It is optimal for the government to keep its initial debt-GDP ratio constant forever and to levy a time-invariant tax rate sufficient to finance a constant ratio of its primary surplus to GDP. The government issues risk-free debt and sells a Shiller security each period. A Bellman equation discounts after-tax, after tax-distortions GDP flowing to households at a rate $r+\lambda-g$ that includes the risk premium λ on the Shiller (3) security in addition to the $r-g$ term that appears prominently in the analysis of Blanchard (14).

We have retained an assumption shared by Arrow (2) and Barro (1) that financial contracts are perfectly enforced and have withheld from government debt an additional "convenience yield" occasionally included in recent positive, as opposed to normative, macro-finance papers. In subsequent work (17), we plan to add features that bring our model closer to observed government debt/GDP series by letting a government default if it is willing to accept consequences; we will also endow government debt with a convenience yield. Adding those features promises to shed light on forces that can impart positive drifts to tax rates and to debt-GDP dynamics, features whose absence is a notable and sometimes counterfactual feature of Barro (1)'s normative model as well as ours.[¶¶]

5. Materials and Methods

A. Arrow Securities Can Replace Shiller's Security. Instead of using Shiller's macro asset to insure GDP shocks, the government can support the same optimal outcomes for the tax rate and risk-free government debt processes by using a complete set of one-time-ahead state-contingent Arrow securities. Either financial arrangement supports outcomes described by Theorem 2.

As noted in Section 1, to obtain $\Pi(\varepsilon_{t+1})$ at time $t+1$ contingent on ε_{t+1} being realized, the government would have to pay $\int \Pi(\varepsilon)m(\varepsilon)d\Phi(\varepsilon)$

[§§] The increasing, convex, and smooth $\theta(\cdot)$ function plays a key role via $-P_B(B_{t+1}, Y_{t+1}) = -p'(b_{t+1}) = -p'(b_0) = 1+\theta'(\tau_0) > 1$.

[¶¶] Studies in applied macroeconomics use a normative model like Barro's to "rationalize" an observed government policy and thereby explain it. Various histories of national fiscal policies have used the Barro (1) model as a benchmark positive model. Sargent (18) compares various normative models as possible explanations of post-WWII inflation history.

at time t. When the government trades Arrow securities, the counterpart to Eq. **6** for the evolution of risk-free government debt $\{B_t\}$ is

$$B_{t+1} = e^r B_t + e^r(\gamma - \tau_t)Y_t - \Pi(\varepsilon_{t+1}) + e^r \int \Pi(\varepsilon)m(\varepsilon)d\Phi(\varepsilon). \quad [30]$$

Substituting Eq. **30** into Eq. **12** gives

$$\begin{aligned} P(B_t, Y_t) = \max_{\tau_t, \Pi}\ & Y_t - \tau_t Y_t - \theta(\tau_t)Y_t \\ & + \mathbb{E}_t\Big[m_{t+1}P\left(e^r B_t + e^r(\gamma - \tau_t)Y_t - \Pi(\varepsilon_{t+1})\right. \\ & \left. + e^r \int \Pi(\varepsilon)m(\varepsilon)d\Phi(\varepsilon), Y_{t+1}\right)\Big]. \end{aligned} \quad [31]$$

Choosing $\Pi(\varepsilon_{t+1})$ for each ε_{t+1} to maximize $P(B_t, Y_t)$ is equivalent to choosing $\Pi(\varepsilon_{t+1})$ for each ε_{t+1} to maximize

$$\begin{aligned} \int \Big[& m(\varepsilon_{t+1})P\left(e^r B_t + e^r(\gamma - \tau_t)Y_t - \Pi(\varepsilon_{t+1})\right. \\ & \left. + e^r \int \Pi(\varepsilon)m(\varepsilon)d\Phi(\varepsilon), Y_{t+1}\right)\Big]d\Phi(\varepsilon_{t+1}). \end{aligned} \quad [32]$$

The first-order necessary condition for Arrow security demand $\Pi(\varepsilon_{t+1})$ in state ε_{t+1} is

$$\begin{aligned} & - m(\varepsilon_{t+1})P_B(B_{t+1}, Y_{t+1})d\Phi(\varepsilon_{t+1}) + [m(\varepsilon)d\Phi(\varepsilon)]\,|_{\varepsilon=\varepsilon_{t+1}} \\ & \int \left[m(\varepsilon_{t+1})e^r P_B(B_{t+1}, Y_{t+1})\right] d\Phi(\varepsilon_{t+1}) = 0. \end{aligned} \quad [33]$$

Once again guessing that $P(B_t, Y_t) = p(b_t)Y_t$ and that $b_{t+1} - b_t = 0$ for all t to simplify Eq. **33**, we confirm that $1 = \mathbb{E}_t\left[m_{t+1}e^r\right]$.

First-order necessary condition for τ_t agrees with Eq. **14**. Combining this outcome with $b_{t+1} - b_t = 0$ for all t, we obtain Eq. **16** for τ_t. In conjunction with budget constraint Eq. **30**, we obtain tax-smoothing result given in Eq. **21** To verify $b_{t+1} - b_t = 0$ for all $t \geq 0$, we can show that for all ε_{t+1}:[##]

$$\begin{aligned} \frac{B_{t+1}}{Y_{t+1}} &= \frac{e^r B_t + e^r(\gamma - \tau_t)Y_t - \Pi(\varepsilon_{t+1}) + e^r \int \Pi(\varepsilon)m(\varepsilon)d\Phi(\varepsilon)}{Y_{t+1}} \\ &= \frac{B_t}{Y_t}. \end{aligned} \quad [34]$$

Data, Materials, and Software Availability. There are no data underlying this work.

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[##] After substituting the tax policy Eq. **21** into Eq. **34**, we obtain:

$$\begin{aligned} & \frac{e^{r-\delta}B_t - \Pi(\varepsilon_{t+1}) + e^r \int \Pi(\varepsilon)m(\varepsilon)d\Phi(\varepsilon)}{Y_t \exp\left[g - \frac{1}{2}\sigma^2 + \sigma\varepsilon_{t+1}\right]} = \frac{B_t}{Y_t} \\ & \Leftrightarrow \left(e^{r-\delta} - e^{g - \frac{1}{2}\sigma^2 + \sigma\varepsilon_{t+1}}\right)B_t = \Pi(\varepsilon_{t+1}) - e^r \int \Pi(\varepsilon)m(\varepsilon)d\Phi(\varepsilon) \\ & \Leftrightarrow \left(-e^{-\delta}R(\varepsilon_{t+1})B_t\right) - e^r \int \left(-e^{-\delta}R(\varepsilon)B_t\right)m(\varepsilon)d\Phi(\varepsilon) \\ & = \Pi(\varepsilon_{t+1}) - e^r \int \Pi(\varepsilon)m(\varepsilon)d\Phi(\varepsilon), \end{aligned}$$

which implies

$$\Pi(\varepsilon_{t+1}) = -e^{-\delta}R(\varepsilon_{t+1})B_t = \Delta_t R(\varepsilon_{t+1})S_t,$$

where the second equality uses the expression for Δ_t given in Eq. **22**.

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