

Internal Capital Markets and Spinoffs: A Dynamic Contracting Approach*

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Abstract

We develop a theory of internal capital markets in which financing and organizational form arise from one optimal dynamic contract. Financing two correlated divisions jointly lowers the volatility of the manager's promised value, delivering incentive-compatible coinsurance. This benefit can come at the cost of joint liquidation and of the inefficiency of operating a conglomerate. Whether to centralize turns on one statistic, risk-(and-agency)-adjusted productivity. A spinoff option makes breakup state-contingent. Under financial distress, the conglomerate sheds the division with lower risk-adjusted productivity to survive; when financially sound, it spins off the less productive division to refocus. The two motives generate two disjoint spinoff regions, one for risk management and the other for efficiency.

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1 Introduction

Multi-division firms produce about three quarters of U.S. manufacturing output (Maksimovic and Phillips, 2002) and they finance their divisions through an internal capital market: headquarters raises funds from outside investors and allocates them inside the firm. Whether this arrangement creates or destroys value is a classic question of corporate finance. The bright side is an informed headquarters that picks winners, monitors divisions, redeploys assets, and pools imperfectly correlated cash flows to diversify risk (Lewellen, 1971; Gertner, Scharfstein, and Stein, 1994; Stein, 1997). The dark side is that the same pooling invites rent seeking and internal power struggles that prop up weak divisions at the expense of strong ones (Rajan, Servaes, and Zingales, 2000; Scharfstein and Stein, 2000). Schoar (2002) finds both sides in the same plant-level data. Decades of empirical work have not settled which side dominates.¹

Both sides of this debate share a premise: the trade-off is settled once, by the firm's primitives. But a conglomerate's financial condition is not a primitive. It moves with its divisions' performance, and it moves the trade-off with it. The value of pooling matters most when survival is at stake. Its cost is paid every period the divisions stay together. Which side dominates therefore depends on where the firm stands, and the organizational response depends on it too: keep the divisions together, or let one go, and which one. These are dynamic questions, and they call for a dynamic setting.

This paper provides one such setting. We study a conglomerate whose financing, payout, liquidation, and organizational form are all governed by one optimal long-term contract. One state variable drives everything: the manager's promised value, which measures the firm's financial slack (DeMarzo and Sannikov, 2006; Biais, Mariotti, Plantin, and Rochet, 2007; DeMarzo and Fishman, 2007). Which side of its internal capital market dominates, whether the conglomerate stays whole, and which division it spins off all depend on the firm's distance to financial distress. The answer reverses as that distance evolves. In the data, distress-driven and refocusing divestitures are documented as distinct phenomena, occurring at different firms.² In our model, both emerge from one spinoff policy, as two disjoint regions of the state space.

We develop the analysis in three steps. Each adds one ingredient and delivers one main result. The first step (Sections 2 and 3) extends the contracting framework of DeMarzo and Sannikov (2006) to two projects with correlated cash flows and analyzes the investor's time-0 choice between decentralized and centralized financing. The result is a bright side of internal

¹See, e.g., Lang and Stulz (1994), Berger and Ofek (1995), Lamont (1997), Shin and Stulz (1998), Ozbas and Scharfstein (2010), Matvos and Seru (2014), and Giroud and Mueller (2015). Maksimovic and Phillips (2013) provide a survey of this literature.

²See, e.g., Asquith, Gertner, and Scharfstein (1994), Comment and Jarrell (1995), John and Ofek (1995), and Lang, Poulsen, and Stulz (1995).

capital markets derived from the optimal contract, together with a sufficient statistic for when centralization is worth it. The second step (Section 5, using the extended model set up once in Section 4) lets the conglomerate spin off either division at any time, an important real-world divestiture option. The result is a theory of spinoffs as survival decisions. The third step (Section 6, the same extended model with $\kappa > 0$) lets keeping two divisions together carry the dark-side cost of internal capital markets. The result is that the direction of the spinoff reverses with the firm's financial condition, in line with the divestiture evidence cited above.

In the first step, a deep-pocketed investor finances a financially constrained manager. The manager privately observes the projects' cash flows and can divert them for private benefit. The optimal contract specifies the manager's compensation, the firm's payouts, and liquidation as functions of a single state variable: the manager's promised value, his continuation value under the contract. At the outset, the investor chooses between decentralized financing, with a stand-alone contract for each project, and centralized financing, with a single conglomerate contract written on both cash-flow streams. Under centralized financing, the investor also chooses how sensitively the manager's promised value responds to each division's cash-flow news. This pair of pay-for-performance sensitivities is at the center of our analysis. In this first step the organizational choice is made once, at inception, and a conglomerate has a single firm-wide termination option.

Our first main result is that the bright side of the internal capital market is derived from the optimal contract. The contracting problem itself is new. In the single-project benchmark of DeMarzo and Sannikov (2006), the manager's no-underreporting incentive compatibility (IC) constraint always binds. By contrast, with two projects it need not. This is because the manager can also divert one division's cash flow to fund overreporting of the other's. Ruling out every such cross-project strategy disciplines the two pay-for-performance sensitivities jointly. Following DeMarzo and Sannikov (2006), we define each division's agency measure ϕ as its cash-flow volatility σ multiplied by its managerial diversion parameter λ . We show that while the higher- ϕ division's no-underreporting constraint always binds, the other division's can fall slack—a feature new to multi-project contracting. When slack, that division's cash flow serves as a hedge: a signal for filtering risk out of the manager's promised value, in the spirit of relative performance evaluation (Holmström, 1979, 1982).

Mathematically, our contracting analysis boils down to a linear-quadratic problem: minimizing the variance of the manager's promised value, ϕ_c^2 , within a convex set formed by linear IC constraints. The correlation between the two divisions' cash flows plays a crucial role (Section 3.3). With positive or mildly negative correlation, both no-underreporting constraints bind. With sufficiently negative correlation, one falls slack as the hedging role above takes over. With extreme negative correlation, the binding concern flips to no-overreporting: the hedging demand itself has to be capped.

This joint determination of pay-for-performance sensitivities at the divisional level is a source of the bright side of internal capital markets: one division's losses are met out of the other's profits before they threaten the firm's survival. Cross-subsidization, which is a defining *ex post* activity of an internal capital market, thus arises as an optimal feature of the contract. It delivers the coinsurance of Lewellen (1971), but earned through incentive compatibility rather than posited.

The time-0 comparison then collapses into a sufficient statistic: the conglomerate's risk-and-agency-adjusted productivity $\eta_c = \mu_c/\phi_c$, the sum of the two divisions' expected cash flows (μ_c) divided by the volatility of the manager's promised value (ϕ_c). For brevity, we refer to η simply as risk-adjusted productivity from here on. Centralized financing is optimal at inception if and only if diversification raises η_c above an endogenous cutoff. The cost of centralization is bundled survival: one contract means one promised-value process for the manager, so liquidation, when it comes, claims both divisions.

In the second step, we ask whether the conglomerate can keep the financing benefit of centralization without accepting joint liquidation. We give it a spinoff option: at any time, either division can be spun off as a stand-alone firm under its own contract. The option need not be exercised, so centralized financing with the option weakly dominates decentralized financing (Section 5). We show that the conglomerate spins off a division only under financial distress, when the manager's promised value falls to an endogenous threshold. It always sheds the division with the lower risk-adjusted productivity $\eta = \mu/\phi$. In this $\kappa = 0$ benchmark, spinoff is a survival decision, not an efficiency decision. The conglomerate does not shed a division to run leaner. It sheds one to keep the other alive, at the point where cross-subsidizing the outgoing division is no longer worth the risk it imposes on the survivor.

In the $\kappa = 0$ benchmark, however, a financially healthy conglomerate never breaks up. How do we reconcile this with the evidence that healthy firms undertake refocusing divestitures (Comment and Jarrell, 1995; John and Ofek, 1995)? In the third step, we introduce a cost of staying diversified. We capture this dark side with a single parameter, κ , the flow cost of running a conglomerate. It is our reduced-form stand-in for the coordination and complexity costs of running two businesses under one roof, the dilution of divisional monitoring and managerial focus, and the rent seeking and internal politicking that pooled resources invite.

Our third main result follows. With $\kappa > 0$, the conglomerate has a second reason to shed a division: not to survive, but to escape the ongoing cost of running two divisions under one roof. The two motives need not agree. Survival favors the division with the higher risk-adjusted productivity. Cost saving favors the one with the higher expected cash flow. When they conflict, the direction of the spinoff reverses over the firm's life cycle, through two disjoint spinoff regions (Section 6). Under financial distress, the firm sheds the lower risk-adjusted-productivity division, as in the benchmark. Once slack recovers, it sheds the less productive

division instead, possibly the very one it protected under distress. The same trade-off decides whether the conglomerate is formed at all. A high enough dark-side cost makes separate financing optimal outright. A moderate one raises the bar that risk-adjusted productivity must clear. Forming and breaking up a conglomerate are two faces of the same trade-off.

Taken together, the model delivers the two kinds of divestiture the empirical literature documents, distress-driven and refocusing, as one state-contingent policy of a single firm. A conglomerate that sheds one division under duress and a different one later, from a position of strength, is not two puzzles requiring two explanations. It is following one optimal contract as its financial slack traverses two spinoff regions.

Related literature. Our paper is related to both the dynamic contracting literature and the literature on internal capital markets. The theories of internal capital markets are primarily static, as discussed above (see Stein (2003) for a survey). Building on the classic two-period contracting model of Bolton and Scharfstein (1990), with binary cash flows and a threat of termination that disciplines a manager who can divert cash flow, Inderst and Müller (2003) extend Bolton and Scharfstein’s analysis to a setting with two identical projects and compare centralized with decentralized financing. That choice, however, is still made once and for all.³ By contrast, that decision in our model is dynamic and state-contingent.

Also related to ours is Leland (2007), who develops a static trade-off model (with tax benefits and distress costs) in which merging two activities pools imperfectly correlated cash flows and supports more debt, but forfeits the separate default options of limited liability. Ours differs in three respects: (a) pooling pays off by lowering the volatility of the manager’s promised value rather than by raising the tax shield; (b) the organizational-form decision is dynamic and state-contingent rather than once-and-for-all; and (c) what the firm sheds, not merely whether it separates, reverses as the conglomerate’s financial slack evolves.

Our contracting environment builds on the dynamic moral hazard models of DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007), and Biais et al. (2007).⁴ Malenko (2019) develops a dynamic contracting model in which headquarters allocates capital to privately informed division managers.⁵ A CEO repeatedly allocates capital to an empire-building division manager who privately observes investment opportunities. The optimal mechanism is implemented through a dynamic capital budget with limited managerial discretion. His contract governs how capital moves inside a firm whose boundary is fixed. Ours governs the

³Using a two-period setting similar to Inderst and Müller (2003)’s, Faure-Grimaud and Inderst (2005) show that a divisional manager can be entrenched in a conglomerate under optimal contracting.

⁴See also Sannikov (2008), He (2009), Hoffmann and Pfeil (2010), Piskorski and Tchisty (2010), Biais et al. (2010), DeMarzo et al. (2012), Zhu (2013), Piskorski and Westerfield (2016), Miao and Rivera (2016), and Feng and Westerfield (2021). Sannikov (2013) surveys this literature.

⁵See, e.g., Harris, Kriebel, and Raviv (1982), Antle and Eppen (1985), and Harris and Raviv (1996, 1998) for static mechanism-design-based capital-budgeting models.

boundary itself: whether the two projects share a contract, and when and which one leaves it. The two papers are complementary. Our two-project contracting problem is related to the multidimensional agency model of Szydlowski (2019), in which the agent allocates effort across projects and the boundary of the firm is fixed.

More broadly, the recursive formulation of repeated moral hazard, with promised value as the state variable, goes back to Green (1987), Spear and Srivastava (1987), and Thomas and Worrall (1990). Albuquerque and Hopenhayn (2004), Quadrini (2004), and Clementi and Hopenhayn (2006) use it to derive borrowing constraints, investment, and exit from an optimal long-term contract. Ljungqvist and Sargent (2018) provide a textbook treatment.

Our contracting model features multidimensional screening, which is notoriously difficult in static settings.⁶ In our dynamic setting, continuous-time martingale representation reduces the choice of incentive sensitivities to a quadratic minimization over a convex set defined by the IC constraints. This yields a closed-form characterization of optimal incentive sensitivities and identifies when one project's cash flow serves as a signal for contracting on the other.

Finally, our paper relates to dynamic models of multi-division firms. In Gomes and Livdan (2004), decreasing returns in the core business drive diversification and later refocusing. Matvos and Seru (2014) estimate a structural model in which internal capital markets substitute for disrupted external ones. In Dai et al. (2024), a two-division firm facing costly external finance sells a division to raise funds when liquidity runs low. We differ from these papers on three counts. First, our financing constraint is endogenous: it arises from moral hazard rather than from costly external finance. Second, organizational form is endogenous throughout, chosen at inception and revised as the manager's promised value evolves. Third, a spinoff in our model is a survival or a refocusing decision, not a source of funds. It therefore occurs in two disjoint regions of the state space, with a different division shed in each, rather than in a single liquidity-scarce region.

The remainder of the paper proceeds as follows. Section 2 sets up the two-project contracting environment. Section 3 derives the optimal contract under centralized financing and the baseline $t = 0$ organizational-choice cutoff. Section 4 sets up the extended model once, with both the spinoff option and the cost κ . Section 5 analyzes it with $\kappa = 0$ and develops our theory of spinoffs. Section 6 turns to $\kappa > 0$ and shows how the dark-side cost generates spinoff reversal and reshapes the $t = 0$ organizational choice. Section 7 concludes.

⁶See, e.g., Rochet and Stole (2003), Rochet (2024), and Daskalakis, Deckelbaum, and Tzamos (2017).

2 Contracting Model with Two Correlated Projects

When multiple projects are bundled under a single contract, the investor can condition managerial compensation on several cash-flow streams at once, turning incentive provision from the one-dimensional problem in DeMarzo and Sannikov (2006) into a multidimensional risk-allocation problem. The manager's continuation value is exposed to a portfolio of project-level shocks, and its volatility governs the risk of inefficient termination: centralized financing can lower that volatility, and hence termination risk, through optimal incentive loadings across projects. But bundling also ties the two projects' survival to a single continuation-value process, so termination occurs jointly rather than independently. The organizational choice therefore trades off diversification benefits against the loss of termination flexibility.

We now formalize this environment. Subsection 2.1 presents the baseline model. Subsection 2.2 formulates the conglomerate's contracting problem under centralized financing. Subsection 2.3 then states the investor's decision rule for choosing between centralized and decentralized financing.

2.1 Setup

Consider a principal-agent setting in which a deep-pocketed investor has two potential projects, a and b , that can be delegated to a manager. At time 0, the investor can finance these projects under one of the following two organizational structures. One option is to set up a single firm and design an optimal contract for the manager to operate both projects. We refer to this organizational structure as a *conglomerate*, denoted by firm c . The other option is to establish two stand-alone (focused) single-project firms, firm a and firm b , which manage projects a and b separately. We refer to the former arrangement as *centralized financing* and the latter as *decentralized financing*.

The manager is talented in operating the projects but is penniless. He is risk neutral and protected by limited liability, meaning that the investor cannot require the manager to make any payment at any time t . An agency problem arises because the manager has an informational advantage over the investor in the operation of the projects.

We extend the one-project setting of DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007) to a two-project environment. As in the existing literature, each project generates independently and identically distributed (*i.i.d.*) cash flows over time. Unlike the one-project benchmark, the cash flows of the two projects are allowed to be correlated. Specifically, we assume that the cumulative cash flows generated under the manager's operation,

$\{(Y_t^a, Y_t^b); t \geq 0\}$, evolve according to

$$dY_t^i = \mu_i dt + \sigma_i dZ_t^i, \quad i = a, b, \quad (1)$$

where $\mu_i > 0$ and $\sigma_i > 0$ are the expected cash-flow rate and volatility coefficient of project i , respectively. The processes $\{(Z_t^a, Z_t^b); t \geq 0\}$ form a bivariate Brownian motion with constant correlation coefficient $\rho \in (-1, 1)$ between Z_t^a and Z_t^b , which drive the cash flows of projects a and b , respectively.⁷ As we later demonstrate, this correlation plays an important role in our analysis.

When managing project i , the manager observes the cash flows $\{Y_t^i; t \geq 0\}$, whereas the investor does not. The manager may divert cash flows for private benefit by reporting $\{\widehat{Y}_t^i; t \geq 0\}$ instead. The discrepancy between $\{Y_t^i\}$ and $\{\widehat{Y}_t^i\}$ arises from the manager's misreporting behavior and constitutes the source of the agency problem.

If the manager underreports project i 's cash flow such that $dY_t^i - d\widehat{Y}_t^i > 0$, he obtains a fraction $\lambda_i \in (0, 1]$ of the diverted amount, namely $\lambda_i(dY_t^i - d\widehat{Y}_t^i)$. When $\lambda_i < 1$, diversion entails deadweight losses. If instead the manager overreports project i 's cash flow so that $dY_t^i - d\widehat{Y}_t^i < 0$, he must inject funds equal to $d\widehat{Y}_t^i - dY_t^i$ to sustain the inflated report.

As in DeMarzo and Sannikov (2006), we focus on a setting in which the manager has no access to an outside savings account.⁸ Without external savings, the manager of a focused firm cannot overreport the cash flows of the project he manages. Similarly, the manager of a conglomerate cannot overreport the cash flows of both projects simultaneously. That is, in a conglomerate, the manager must (weakly) underreport at least one project's cash flow. Without loss of generality, suppose that the manager underreports project i , so that $dY_t^i > d\widehat{Y}_t^i$. Using the diverted funds $\lambda_i(dY_t^i - d\widehat{Y}_t^i)$, he can increase his consumption dU_t , finance the overreporting of the other project's cash flow, or choose a combination of the two.

If the manager underreports project i and uses the entire proceeds to finance the overreporting of project $-i$, the following condition must hold:

$$d\widehat{Y}_t^{-i} - dY_t^{-i} = \lambda_i(dY_t^i - d\widehat{Y}_t^i) > 0. \quad (2)$$

More generally, the manager may underreport both projects and divert funds for his own

⁷Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an information filtration $\{\mathcal{F}_t; t \geq 0\}$ generated by the Brownian motion (Z_t^a, Z_t^b) with covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. The filtration $\mathcal{F}_t$ contains all $\mathbb{P}$ -null sets of $\mathcal{F}$ for all $t \geq 0$. We take $\rho \in (-1, 1)$, which keeps this covariance matrix nonsingular and the analysis tight. We nonetheless refer to the perfectly correlated cases $\rho = 1$ and $\rho = -1$. These can be analyzed either directly or as the one-sided limits $\rho \uparrow 1$ and $\rho \downarrow -1$, holding the promised value and all other primitives fixed; both routes deliver the same solution. We therefore write $\rho \in (-1, 1)$ in the main analysis and read $\rho = \pm 1$ in this way.

⁸In Appendix A.1, we extend the analysis in DeMarzo and Sannikov (2006) to a setting in which the manager has access to an external savings account.

consumption. Given any reporting strategy, the manager receives the following nonnegative consumption flow over the interval dt :

$$dC_t = dU_t + [dY_t^a - d\hat{Y}_t^a]^{\lambda_a} + [dY_t^b - d\hat{Y}_t^b]^{\lambda_b}, \quad (3)$$

where

$$[dY_t^i - d\hat{Y}_t^i]^{\lambda_i} = \lambda_i \max\{dY_t^i - d\hat{Y}_t^i, 0\} - \max\{d\hat{Y}_t^i - dY_t^i, 0\}. \quad (4)$$

The first term in (4) captures the net proceeds from underreporting project i 's cash flow, adjusted by the diversion efficiency parameter λ_i . The second term represents the net outflow required to support overreporting.

We next characterize the investor's problem under centralized financing.

2.2 Contracting for the Conglomerate: Centralized Financing

The investor offers the manager a long-term contract that specifies a nondecreasing cash-compensation process $\{U_t\}$ and a contract-termination time τ , both contingent on the manager's reported cash-flow processes $\hat{\mathbf{Y}} := \{(\hat{Y}_t^a, \hat{Y}_t^b); t \geq 0\}$.⁹ The manager is risk neutral and discounts his consumption at a rate $\gamma > 0$. We normalize both the firm's liquidation value and the manager's outside option to zero.¹⁰

Given a contract $(\{U_t\}, \tau)$, the manager chooses a reporting strategy $\hat{\mathbf{Y}}$ to maximize his expected utility from consumption dC_t :

$$W_0 = \mathbb{E} \left[\int_0^\tau e^{-\gamma t} dC_t \right]. \quad (5)$$

For a given contract $(\{U_t\}, \tau)$, the manager's reporting strategy $\hat{\mathbf{Y}}$ is said to be *incentive compatible* if it maximizes his expected payoff W_0 in (5). An incentive-compatible contract is therefore a triplet $(\{U_t\}, \tau, \hat{\mathbf{Y}})$ consisting of a contract and an associated reporting strategy $\hat{\mathbf{Y}}$ that the investor recommends to the manager.

As in DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007), the investor discounts cash flows at a rate $r > 0$. Following the literature, we assume $\gamma > r$, so that the manager is more impatient than the investor.¹¹ This assumption implies that, absent incentive concerns, the investor prefers to compensate the manager sooner rather than later and rules out contracts that indefinitely postpone payments.

⁹Formally, the admissible set of the investor's strategy $(\{U_t\}, \tau)$ is adapted to the filtration $\{\hat{\mathcal{F}}_t\}$ generated by $\hat{\mathbf{Y}}$.

¹⁰See Appendix A.2 for details.

¹¹This impatience may be preference-based or may arise indirectly if the manager has access to more attractive alternative investment opportunities.

The investor's optimal contracting problem is to design an incentive-compatible contract $(\{U_t\}, \tau, \hat{\mathbf{Y}})$ that maximizes his expected payoff:

$$\mathbb{E} \left[\int_0^\tau e^{-rt} (d\hat{Y}_t^a + d\hat{Y}_t^b) - \int_0^\tau e^{-rt} dU_t \right], \quad (6)$$

subject to delivering the manager an initial promised payoff W_0 defined in (5).

Let $P_{c,0}$ denote the value of (6) under the optimal contract, and let $P_c(W_0)$ denote the corresponding investor value function, so that $P_{c,0} = P_c(W_0)$. Assuming that the investor has all the bargaining power, he chooses W_0 to maximize $P_c(W_0)$ subject to the manager's participation constraint $W_0 \geq 0$. Let

$$W^{c,*} = \arg \max_{W_0 \geq 0} P_c(W_0). \quad (7)$$

2.3 The Organizational Choice: A Decision Rule

Let $P_i(W)$ denote the investor's value function for a stand-alone firm operating a single project $i = a, b$, as in DeMarzo and Sannikov (2006). Let P_i^* denote the investor's value at $t = 0$ for the problem:

$$P_i^* = \max_{W \geq 0} P_i(W), i = a, b. \quad (8)$$

The investor's value under decentralized financing is

$$P_d^* := P_a^* + P_b^* = \max_{W \geq 0} P_a(W) + \max_{W \geq 0} P_b(W). \quad (9)$$

The investor chooses centralized financing if $\max_W P_c(W) > P_d^*$, and decentralized financing if $\max_W P_c(W) < P_d^*$. The investor is indifferent between the two organizational structures when $\max_W P_c(W) = P_d^*$. We discuss the economic trade-off between the two organizational structures later in Subsection 3.4.

3 Optimal Contract and Organizational Choice

In this section, we first characterize the incentive compatibility conditions for a conglomerate in Subsection 3.1, then derive the solution to the conglomerate's optimal contracting problem in Subsections 3.2 and 3.3, and finally analyze the investor's organizational choice between centralized and decentralized financing in Subsection 3.4.

3.1 Incentive Compatibility (IC)

Following DeMarzo and Sannikov (2006), we introduce the manager's continuation-value process $\{W_t\}$. Under the assumption that the manager truthfully reports both projects' cash flows after time t , and given the reporting history $\hat{\mathbf{Y}}^t := \{(\hat{Y}_s^a, \hat{Y}_s^b); 0 \leq s < t\}$, the manager's promised value at time t is defined as

$$W_t(\hat{\mathbf{Y}}^t) = \mathbb{E} \left[\int_t^\tau e^{-\gamma(s-t)} dU_s \middle| \hat{\mathcal{F}}_t \right]. \quad (10)$$

For any incentive-compatible contract, applying the martingale representation theorem to W_t in (10) implies that there exists a pair $(\beta_{a,t}, \beta_{b,t})$, adapted to the filtration $\{\hat{\mathcal{F}}_t\}$, such that for all $t < \tau$,

$$dW_t = \gamma W_t dt - dU_t + \beta_{a,t}(\hat{\mathbf{Y}}^t) (d\hat{Y}_t^a - \mu_a dt) + \beta_{b,t}(\hat{\mathbf{Y}}^t) (d\hat{Y}_t^b - \mu_b dt). \quad (11)$$

Define $\hat{\mathcal{Z}}_t^i := (\hat{Y}_t^i - \mu_i t) / \sigma_i$ for $i = a, b$.¹²

The sensitivity pair $(\beta_{a,t}, \beta_{b,t})$ is determined by the manager's reported cash flows up to time t . The manager's incremental compensation over $(t, t + dt)$ consists of the cash transfer dU_t and the change in promised value dW_t . Because the manager discounts future payoffs at a rate γ , the expected value of $dW_t + dU_t$ must equal $\gamma W_t dt$. The last two terms in (11) capture how the manager's W responds to unexpected project-level performance: the coefficients (β_a, β_b) measure the sensitivity of W_t to the innovations $d\hat{Y}_t^i - \mu_i dt$ for $i = a, b$.

We now characterize the incentive-compatibility constraints. To better understand their structure, it is useful to first clarify the economic logic of the conglomerate's contracting problem. The only state variable is the manager's promised continuation value W_t . Under centralized financing, the investor can adjust how W_t loads on the two project-level cash-flow innovations through the sensitivity pair $(\beta_{a,t}, \beta_{b,t})$. Because incentive provision is costly—and this cost increases with the volatility of the continuation value process—the conglomerate's problem can be viewed as choosing $(\beta_{a,t}, \beta_{b,t})$ to minimize the volatility of W_t subject to incentive-compatibility constraints. Since the instantaneous variance of W_t equals $\beta_{a,t}^2 \sigma_a^2 + \beta_{b,t}^2 \sigma_b^2 + 2\rho \beta_{a,t} \beta_{b,t} \sigma_a \sigma_b$, the investor's problem is to choose a point in the admissible set $\mathcal{C}$ that minimizes this quadratic form.

The key trade-off is therefore geometric. On the one hand, loading W_t on multiple shocks may allow the investor to diversify or partially orthogonalize risk, thereby reducing the volatil-

¹²The pair $\{(\hat{\mathcal{Z}}_t^a, \hat{\mathcal{Z}}_t^b); t \geq 0\}$ forms a bivariate Brownian motion under a probability measure $\hat{\mathbb{P}}$ characterized in Appendix B.1. An equivalent change of measure preserves quadratic covariation, so its covariance matrix is $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, identical to that of $\{(\mathcal{Z}_t^a, \mathcal{Z}_t^b); t \geq 0\}$.

ity of promised value and lowering incentive costs. On the other hand, the manager's ability to misreport project-level cash flows imposes incentive constraints that restrict the feasible sensitivity pair $(\beta_{a,t}, \beta_{b,t})$. As we show below, the optimal contract must lie on the boundary of this two-dimensional IC region: if neither constraint binds, the investor can further reduce volatility without violating incentives. Consequently, the solution reduces to a constrained volatility-minimization problem whose geometry determines the structure of the optimal centralized contract.

Proposition 1. *Truth-telling is incentive-compatible if and only if the following conditions hold for all $t \leq \tau$:*

$$\beta_{i,t} \geq \lambda_i, \quad \text{for } i = a, b; \quad (12)$$

$$\lambda_b \leq \frac{\beta_{b,t}}{\beta_{a,t}} \leq \frac{1}{\lambda_a}. \quad (13)$$

Let $\mathcal{C}$ denote the closed convex set of (β_a, β_b) defined by inequalities (12)–(13).

Condition (12) coincides with the incentive-compatibility constraint in DeMarzo and Sannikov (2006). The novel constraint in our model is (13), which captures the manager's incentive to underreport one project's cash flow in order to obtain funds for overreporting the other project's cash flow. Figure 1 illustrates the admissible convex set of (β_a, β_b) .

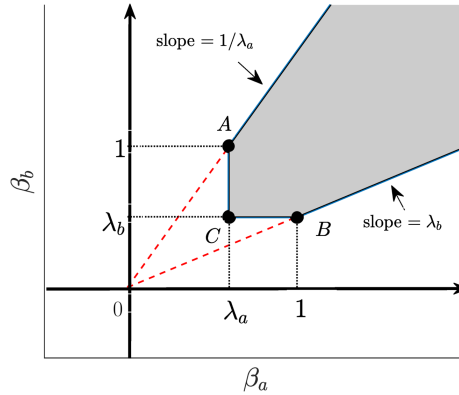


Figure 1: The incentive-compatible set $\mathcal{C}$ of pay-for-performance sensitivities (β_a, β_b) . The shaded area is the closed convex set $\mathcal{C}$ defined by (12)–(13). Its vertices are $C = (\lambda_a, \lambda_b)$, $A = (\lambda_a, 1)$, and $B = (1, \lambda_b)$, and its two unbounded edges lie on the rays through the origin with slopes $1/\lambda_a$ and λ_b .

Proof. We focus on constraint (13), as the proof of (12) follows directly from DeMarzo and Sannikov (2006). Fix a small interval dt and consider a deviation in which the manager diverts δdt from project a 's cash flow to overreport project b 's cash flow, where $\delta > 0$ is constant. The

manager uses $\lambda_a \delta dt$ of diverted funds to finance the overreporting. His continuation value therefore increases by $\beta_b \lambda_a \delta dt$, but decreases by $\beta_a \delta dt$ due to the underreporting of project a . The net gain from this deviation equals $(\beta_b \lambda_a - \beta_a) \delta dt$. To deter such a deviation, the contract must satisfy

$$\beta_b \lambda_a - \beta_a \leq 0 \quad \Longleftrightarrow \quad \frac{\beta_b}{\beta_a} \leq \frac{1}{\lambda_a}. \quad (14)$$

A symmetric argument implies $\frac{\beta_b}{\beta_a} \geq \lambda_b$. Combining these inequalities yields (13). $\square$

As in DeMarzo and Sannikov (2006), the revelation principle implies that there exists an optimal contract under which the manager truthfully reports cash flows. Under truth telling, the manager's continuation value evolves as

$$dW_t = \gamma W_t dt - dU_t + \beta_{a,t} \sigma_a dZ_t^a + \beta_{b,t} \sigma_b dZ_t^b, \quad t < \tau, \quad (15)$$

subject to the terminal condition

$$W_\tau = 0. \quad (16)$$

We are now ready to formulate the conglomerate's contracting problem as a constrained optimization problem.

Problem 1. *The conglomerate's investor chooses $(\{U_t\}, \tau, \beta_a, \beta_b)$ to maximize*

$$P_0 = \mathbb{E} \left[\int_0^\tau e^{-rt} ((\mu_a + \mu_b) dt - dU_t) \right], \quad (17)$$

subject to the IC constraints (12)–(13) and the continuation-value dynamics (15)–(16).

3.2 Investor's Value and General Contracting Properties

In this subsection, we characterize the investor's value function and the general properties of the optimal contract for a conglomerate in two steps.¹³ First, we summarize results that hold in both DeMarzo and Sannikov (2006) and our setting in Section 3.2.1. We then present the new results specific to the conglomerate in Section 3.2.2.

3.2.1 Common Properties in DeMarzo and Sannikov (2006) and Our Setting

We begin by characterizing the conditions for optimal cash compensation. As in DeMarzo and Sannikov (2006), because the manager is more impatient than the investor, it is optimal

¹³To ease exposition, as in DeMarzo and Sannikov (2006), we derive the solution using an intuitive approach in this section. In Appendix E, we formally characterize the optimal contract using a variational-inequality approach and provide a proof.

for the investor to make a lump-sum transfer when the manager's continuation value W_t becomes sufficiently high. Specifically, because an immediate cash payment $dU_t > 0$ is always feasible, the investor's value function must satisfy $P_c(W_t) \geq P_c(W_t - dU_t) - dU_t$, where the right side represents the investor's value if he pays dU_t immediately, thereby reducing the manager's continuation value by dU_t ; it is optimal to pay when this inequality holds with equality.

Taking the limit as $dU_t \rightarrow 0$ yields: $P'_c(W) \geq -1$, which states that the marginal cost of compensating the manager cannot exceed the marginal cost of an immediate payout, normalized to one. Because P_c is concave, there exists a unique threshold $\bar{W}^c$ such that it is optimal to pay the manager a lump-sum transfer whenever $W_t > \bar{W}^c$, bringing his continuation value back to $\bar{W}^c$. Accordingly, the optimal cash-compensation rule is

$$dU_t = \max\{W_t - \bar{W}^c, 0\}, \quad \forall t \geq 0, \quad (18)$$

where $\bar{W}^c$ is characterized by

$$P'_c(\bar{W}^c) = -1. \quad (19)$$

In the cash-compensation region $W > \bar{W}^c$, the investor's value function is linear:

$$P_c(W) = P_c(\bar{W}^c) - (W - \bar{W}^c). \quad (20)$$

Optimality of the payout boundary additionally requires the super-contact condition

$$P''_c(\bar{W}^c) = 0. \quad (21)$$

Second, as in DeMarzo and Sannikov (2006), the lower bound on the manager's continuation value is given by his outside option, which we normalize to zero:

$$P_c(0) = 0. \quad (22)$$

Because liquidation is inefficient, the optimal liquidation time is

$$\tau_c = \inf\{t \geq 0 : W_t = 0\}. \quad (23)$$

Third, consider the region $W \in (0, \bar{W}^c)$, in which the investor makes no cash compensation and the firm is solvent. This region is where the conglomerate's problem differs materially from a one-project setting.

3.2.2 New Properties for a Conglomerate

Under the manager's truthful reporting of the projects' cash flows, (15) implies that the manager's continuation value evolves as

$$dW_t = \gamma W_t dt + \beta_{a,t} \sigma_a dZ_t^a + \beta_{b,t} \sigma_b dZ_t^b, \quad (24)$$

for a given pair $(\beta_{a,t}, \beta_{b,t}) \in \mathcal{C}$. Using dynamic programming, we obtain the following Hamilton–Jacobi–Bellman (HJB) equation for $P_c(W)$ in the region $W \in (0, \bar{W}^c)$:

$$rP_c(W) = \max_{(\beta_a, \beta_b) \in \mathcal{C}} \frac{h(\beta_a, \beta_b)}{2} P_c''(W) + \gamma W P_c'(W) + \mu_a + \mu_b, \quad (25)$$

where $h(\beta_a, \beta_b)$ is quadratic in (β_a, β_b) :

$$h(\beta_a, \beta_b) = \sigma_a^2 \beta_a^2 + 2\rho \sigma_a \sigma_b \beta_a \beta_b + \sigma_b^2 \beta_b^2. \quad (26)$$

Lemma 1. *The investor's value function $P_c(W)$ is globally concave.*

The investor's value function $P_c(W)$ is concave because the investor is endogenously averse to inefficient termination. We provide a proof for the concavity of $P_c(W)$ in Appendix E. This result extends the concavity of the value function for a one-project firm in DeMarzo and Sannikov (2006) to a two-project setting. Importantly, global concavity allows us to characterize the optimal contract in closed form. Since $P_c''(W) < 0$ in (25) and (β_a, β_b) enters only through the diffusion term and the convex set $\mathcal{C}$ defined in (12) and (13), the optimal pair (β_a^*, β_b^*) uniquely solves the following static constrained linear-quadratic minimization problem.

Problem 2. *Choose $(\beta_a, \beta_b) \in \mathcal{C}$, where $\mathcal{C}$ is the IC region defined by (12)–(13), to minimize the quadratic function $h(\beta_a, \beta_b)$ given in (26).*

Below, to solve **Problem 2**, we introduce $\phi_i = \lambda_i \sigma_i$. Intuitively, the larger λ_i is or the more volatile project i is, the larger the agency cost at the project level. This measure of agency costs is motivated by DeMarzo and Sannikov (2006), to which we turn next.¹⁴ Let ϕ_c denote the volatility of the manager's promised value under the optimal contract:

$$\phi_c := \sqrt{h(\beta_a^*, \beta_b^*)}. \quad (27)$$

Note that the project's cash-flow volatility σ and λ always appear multiplicatively through $\phi = \lambda \sigma$ in the optimal contract for a one-project firm (DeMarzo and Sannikov, 2006). In

¹⁴To obtain the solution for a one-project firm, without loss of generality, we set $\mu_b = \sigma_b = 0$. For notational brevity, we omit all superscripts and subscripts for a one-project firm.

contrast, they generally enter separately in our conglomerate setting. This is because the pay-for-performance sensitivity for one project affects not only how the manager's promised value responds to that project's cash-flow shock, but also the investor's choice of the sensitivity for the other project. We next characterize the investor's value function and the optimal contract for a conglomerate.

Theorem 1. *Under the optimal contract, the following results hold: (a) the optimal pay-for-performance sensitivity pair (β_a^*, β_b^*) is the unique solution of **Problem 2** and is time-invariant; (b) the investor's value function $P_c(W)$ and the optimal cash-compensation (payout) threshold $\bar{W}^c$ uniquely solve the ODE*

$$rP_c(W) = \frac{\phi_c^2}{2} P_c''(W) + \gamma W P_c'(W) + \mu_a + \mu_b, \quad 0 < W < \bar{W}^c, \quad (28)$$

subject to the boundary conditions (19), (20), (21), and (22); (c) termination occurs at the first-hitting time τ_c defined in (23); (d) only when the manager's promised value W_t is sufficiently high, i.e., exceeds the endogenous payout threshold $\bar{W}^c$, does the manager receive cash compensation $dU_t > 0$, given in (18).

See Appendix E for a proof of Theorem 1.

Parts (c) and (d) are inherited from DeMarzo and Sannikov (2006): conditional on $\bar{W}^c$ and (β_a^*, β_b^*) , the termination rule and the cash-compensation policy are as in the one-project problem, with $\bar{W}^c$ pinned down by the smooth-pasting condition (19) and the super-contact condition (21). We now turn to the new results for a conglomerate, namely parts (a) and (b) in Theorem 1.

Condition 1. *Without loss of generality, for the remainder of the paper, let project a be the one with the larger agency cost: $\phi_a \geq \phi_b$.*

Corollary 1. *Under Condition 1, the no-underreporting-for-consumption IC constraint in (12) for the project with the larger agency cost always binds, so that*

$$\beta_a^* = \lambda_a, \text{ if } \phi_a \geq \phi_b. \quad (29)$$

We begin by characterizing the structure of the optimal sensitivity pair. Because the investor's problem reduces to minimizing the quadratic form $h(\beta_a, \beta_b)$ over the convex IC set $\mathcal{C}$, the optimal solution must lie on the boundary of $\mathcal{C}$. If neither no-underreporting-for-consumption constraint in (12) binds, there exists an $s \in (0, 1)$ such that $(s\beta_a, s\beta_b) \in \mathcal{C}$. Since (β_a, β_b) enters the HJB (25) only through the diffusion term and $P_c(W)$ is strictly concave in $(0, \bar{W}^c)$, scaling down the sensitivity pair strictly reduces volatility and increases the investor's value. Hence, at least one constraint in (12) must bind at the optimum.

Because at least one of the two no-underreporting-for-consumption constraints (12) must bind, it is optimal for the investor to let it bind for the project with the greater agency cost, as tightening incentives on the more distortionary project yields the largest reduction in aggregate volatility.

Given $\beta_a^* = \lambda_a$, the remaining object to characterize is the pay-for-performance sensitivity for the project with the smaller agency cost, β_b^* . Because the first term in (26) equals $\sigma_a^2 \lambda_a^2$ under the optimal contract, β_b^* is chosen to minimize the sum of the covariance term $2\rho\sigma_a\sigma_b\beta_a\beta_b$ and the variance term $\sigma_b^2\beta_b^2$, subject to the IC constraints (12) and (13). Depending on parameter values, there are three possible cases: *case ab*, *case a1*, and *case a2*.

Which boundary segment of the IC set $\mathcal{C}$ is optimal depends on the relative magnitude of cross-project diversification benefits and agency costs. Because the covariance term $2\rho\sigma_a\sigma_b\beta_a\beta_b$ in (26) becomes more negative as ρ decreases, sufficiently negative correlation can make it optimal to relax one IC constraint in order to exploit cross-project hedging. When correlation is positive or only moderately negative, however, diversification benefits are limited, and both no-underreporting constraints bind. We now analyze these three cases in turn.

3.3 Three Cases: Case ab, Case a1, and Case a2

3.3.1 Case ab: $-\rho \leq \phi_b/\phi_a$

In this case, ρ is either positive or moderately negative. As a result, the no-underreporting-for-consumption IC constraints for both projects bind:

$$\beta_a^* = \lambda_a \quad \text{and} \quad \beta_b^* = \lambda_b, \quad (30)$$

which implies that the variance of the manager's promised value per unit of time equals $\phi_c^2 = h(\beta_a^*, \beta_b^*) = \phi_a^2 + \phi_b^2 + 2\rho\phi_a\phi_b$.

Equation (30) implies that it is optimal to minimize the exposure of the manager's promised value to project b 's cash-flow shock subject to the IC constraint (12) by setting $\beta_b^* = \lambda_b$ (in addition to $\beta_a^* = \lambda_a$).¹⁵ The intuition for (30) is straightforward. Because correlation is positive or only moderately negative in this case, the covariance term in (26) does not generate sufficiently large diversification benefits. As a result, increasing β_b beyond its IC lower bound λ_b would lower the covariance term only marginally while directly increasing the variance term $\sigma_b^2\beta_b^2$. It is therefore optimal to keep both projects at their minimum incentive levels, $\beta_i^* = \lambda_i$.

¹⁵This result can be seen in two steps. First, $(\beta_a, \beta_b) = (\lambda_a, \lambda_b)$ is locally optimal because (1) $\frac{\partial h}{\partial \beta_b}|_{(\lambda_a, \lambda_b)} = 2\sigma_b(\phi_b + \rho\phi_a) \geq 0$, implied by $\phi_b \geq -\rho\phi_a$ (the condition for case ab), and (2) $\frac{\partial h}{\partial \beta_a}|_{(\lambda_a, \lambda_b)} = 2\sigma_a(\phi_a + \rho\phi_b) > 2\sigma_a(\phi_a - \phi_b) \geq 0$ as $\phi_a \geq \phi_b$ under **Condition 1**. Second, for **Problem 2**, a constrained linear-quadratic optimization problem, the local optimum is unique and hence globally optimal.

To sum up, in case *ab*, the two projects are treated symmetrically, so $\beta_i^* = \lambda_i$, and the corresponding variance of the manager's promised value equals $h(\lambda_a, \lambda_b)$. Using (30), Theorem 1 fully characterizes the remaining components of the optimal contract.

Figure 2 plots the time-invariant variance (per unit of time) of the manager's promised value, $h(\lambda_a, \lambda_b)$, for case *ab*. In general, the level curve of $h(\beta_a, \beta_b)$ through (λ_a, λ_b) is an ellipse (see Panel A). When the two projects are symmetric with independent cash-flow shocks, i.e., $\sigma_a = \sigma_b$ and $\rho = 0$, it becomes a circle (see Panel B).

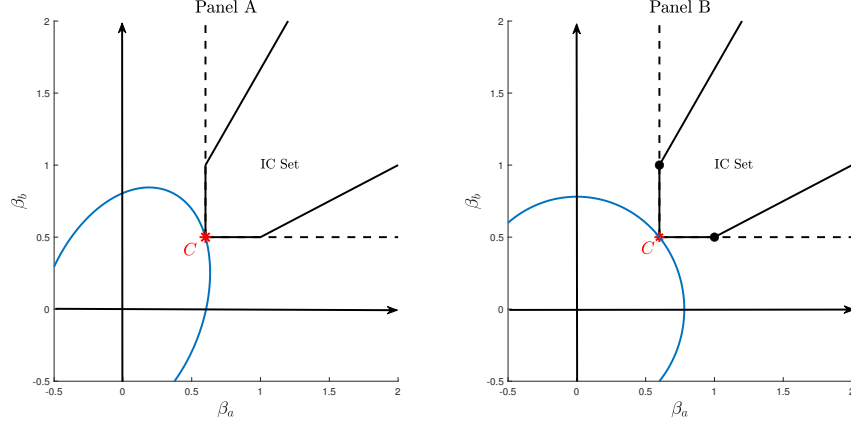


Figure 2: Optimal pay-for-performance sensitivities in case *ab*. In both panels the optimal pair is $(\beta_a^*, \beta_b^*) = (\lambda_a, \lambda_b)$, the vertex C of the IC set. The blue curve is the level curve of $h(\beta_a, \beta_b)$ through C : an ellipse in Panel A and, because $\rho = 0$ and $\sigma_a = \sigma_b$, a circle in Panel B. Parameter values: $\lambda_a = 0.6$, $\lambda_b = 0.5$, and $\sigma_b = 0.6$ in both panels; $\sigma_a = 0.8$ and $\rho = -0.3$ in Panel A; $\sigma_a = 0.6$ and $\rho = 0$ in Panel B.

3.3.2 Case a1: $\phi_b/\phi_a \leq -\rho \leq \sigma_b/\phi_a$

This case captures the economically most interesting regime. Here correlation is sufficiently negative that cross-project diversification benefits become large, but not so large that incentive constraints on project *b* bind. As a result, the investor optimally exploits cash-flow correlation to reduce the volatility of the manager's promised value.

The optimal pay-for-performance sensitivity pair is

$$\beta_a^* = \lambda_a \quad \text{and} \quad \beta_b^* = -\frac{\rho\phi_a}{\sigma_b}. \quad (31)$$

To obtain (31), we first minimize $h(\lambda_a, \beta_b)$ unconstrained and then verify that the resulting minimizer satisfies the IC constraints (12) and (13).

The intuition for (31) is as follows. Because ρ is more negative than in case *ab*, cross-project diversification benefits are larger. Moreover, maximizing diversification subject to $\beta_a^* = \lambda_a$

is fully consistent with the incentive constraints for project b , because the IC constraints for project b do not bind at $\beta_b^* = -\frac{\rho\phi_a}{\sigma_b}$. This is why (31) is optimal in this case.¹⁶

Using (31), we can show that the manager's promised value process in the region $W \in (0, \bar{W}^c)$ evolves as

$$dW_t = \gamma W_t dt + \sqrt{1 - \rho^2} \phi_a dZ_t^\perp. \quad (32)$$

In (32), $Z_t^\perp$ is a Brownian motion that drives the component of project a 's cash-flow shock that is orthogonal to project b 's cash-flow shock. The coefficient $\sqrt{1 - \rho^2} \phi_a$ is the volatility of the manager's promised value per unit of time; it equals $\beta_a^* = \lambda_a$ multiplied by $\sqrt{1 - \rho^2} \sigma_a$, the volatility of the orthogonal component of project a 's cash-flow shock.¹⁷

To sum up, in case a1, the two projects play fundamentally different roles under the optimal contract. Project a , which has the larger agency cost, remains tightly constrained with $\beta_a^* = \lambda_a$. Project b , in contrast, is used as an informational instrument: its cash-flow realization allows the investor to filter out the component of project a 's shock that is spanned by project b . Because the IC constraints for project b are slack at $\beta_b^* = -\frac{\rho\phi_a}{\sigma_b}$, the investor can fully exploit cross-project correlation to minimize volatility.

Consequently, the manager's continuation value depends only on the orthogonal component $dZ_t^\perp$, as if the firm consisted solely of project a but with an effectively reduced volatility $\sqrt{1 - \rho^2} \sigma_a$. Centralization thus reduces the volatility of the manager's promised value by eliminating the spanned component of risk.

3.3.3 Case a2: $-\rho > \sigma_b/\phi_a$

In this case, correlation is so negative that the unconstrained minimizer of $h(\lambda_a, \beta_b)$ violates the no-overreporting IC constraint for project b . The optimal pay-for-performance sensitivity pair is thus

$$\beta_a^* = \lambda_a \quad \text{and} \quad \beta_b^* = 1. \quad (33)$$

Recall that the value of β_b that globally minimizes $h(\lambda_a, \beta_b)$ subject to $\beta_a^* = \lambda_a$ is $\beta_b = -\frac{\rho\phi_a}{\sigma_b} > 1$. However, this choice is infeasible in case a2 because $\beta_b > 1$ violates the no-overreporting IC constraint in (13) for project b . Equivalently, the pair $(\beta_a, \beta_b) = (\lambda_a, -\frac{\rho\phi_a}{\sigma_b})$ lies outside the IC set $\mathcal{C}$. The investor therefore reduces β_b until (λ_a, β_b) reaches a vertex of

¹⁶To show this result, we use $\frac{\partial h}{\partial \beta_b} \big|_{(\lambda_a, -\frac{\rho\phi_a}{\sigma_b})} = 2\sigma_b(-\rho\phi_a + \rho\phi_a) = 0$, $\frac{\partial h}{\partial \beta_a} \big|_{(\lambda_a, -\frac{\rho\phi_a}{\sigma_b})} = 2\sigma_a(1 - \rho^2)\phi_a > 0$, and the fact that for **Problem 2**, local optimality is global optimality, as discussed for case ab. As in case ab, the no-overreporting IC constraint does not bind.

¹⁷We can decompose project a 's cash-flow shock as $dZ_t^a = \rho dZ_t^b + \sqrt{1 - \rho^2} dZ_t^\perp$. By construction, $Z_t^\perp$ is a Brownian motion defined by $Z_t^\perp = \frac{Z_t^a - \rho Z_t^b}{\sqrt{1 - \rho^2}}$, and it is orthogonal to Z_t^b .

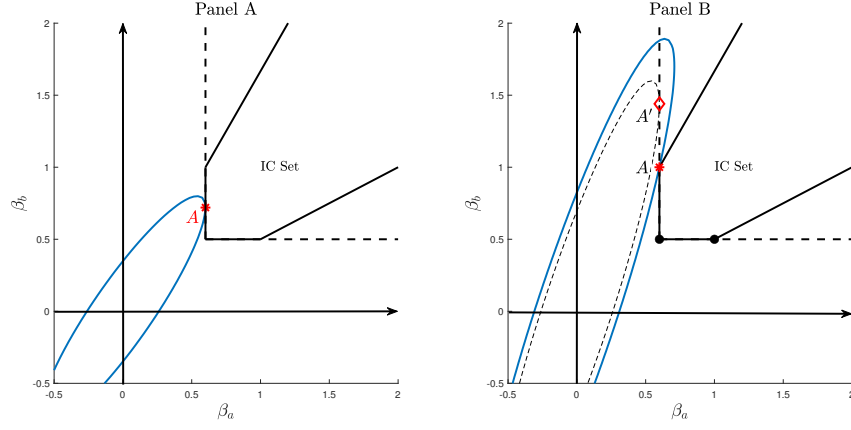


Figure 3: Optimal pay-for-performance sensitivities in cases a1 and a2. In Panel A (case a1), minimizing h over β_b given $\beta_a = \lambda_a$ yields the red star $A = (\lambda_a, -\rho\phi_a/\sigma_b)$, which lies on the boundary of the IC set. In Panel B (case a2), the same minimizer, A' (open diamond), lies outside the IC set, so the optimum is the vertex $A = (\lambda_a, 1)$ (red star); the solid and dashed curves are the level curves of h through A and A' . Parameter values: $\lambda_a = 0.6$, $\lambda_b = 0.5$, $\sigma_a = 0.8$, and $\rho = -0.9$ in both panels; $\sigma_b = 0.6$ in Panel A and $\sigma_b = 0.3$ in Panel B.

$\mathcal{C}$, namely vertex A in Panel B of Figure 3, with coordinates given by (33). The manager's continuation value then evolves as

$$dW_t = \gamma W_t dt + \phi_a dZ_t^a + \sigma_b dZ_t^b, \quad W \in (0, \bar{W}^c), \quad (34)$$

with volatility coefficient $\sqrt{h(\beta_a^*, \beta_b^*)} = \sqrt{\phi_a^2 + \sigma_b^2 + 2\rho\phi_a\sigma_b}$.

Note that in case a2, the no-underreporting-for-consumption IC constraint (12) binds for project a , the project with larger ϕ , while the no-overreporting IC constraint (13) binds for project b . Intuitively, when cross-project diversification benefits are large (i.e., ρ is sufficiently negative), the investor must sacrifice some diversification benefits by capping β_b at 1 in order to control the manager's incentive to overreport project b 's cash flow.

3.3.4 Solution Summary

We can now summarize the volatility of the manager's promised value under centralized financing across the three cases. Under **Condition 1**, that is, $\phi_a \geq \phi_b$, the variance of the manager's promised value per unit of time, $h(\beta_a^*, \beta_b^*)$, is given by

$$h(\beta_a^*, \beta_b^*) = \begin{cases} \phi_a^2 + 2\rho\phi_a\phi_b + \phi_b^2, & \text{if } \rho \geq -\frac{\phi_b}{\phi_a} \quad (\text{case ab}), \\ (1 - \rho^2)\phi_a^2, & \text{if } -\frac{\sigma_b}{\phi_a} \leq \rho \leq -\frac{\phi_b}{\phi_a} \quad (\text{case a1}), \\ \phi_a^2 + 2\rho\phi_a\sigma_b + \sigma_b^2, & \text{if } \rho \leq -\frac{\sigma_b}{\phi_a} \quad (\text{case a2}). \end{cases} \quad (35)$$

$$(36)$$

$$(37)$$

Figure 4 plots $h(\beta_a^*, \beta_b^*)$ as a function of ρ . The function is increasing in ρ : more negative correlation reduces volatility under centralized financing. If $\sigma_b < \phi_a$, all three cases arise. Determining the relevant case therefore requires identifying the range to which the correlation coefficient ρ belongs (see Panel A and equation (35)). If instead $\sigma_b > \phi_a$, only cases ab and a1 exist.¹⁸ Intuitively, case a2 cannot arise in this situation because project b 's volatility is sufficiently large that the no-overreporting IC constraint for project b does not bind; in particular, the optimal sensitivity satisfies $\beta_b^* < 1$.

Unlike in the one-project setting, in the two-project setting the principal can reduce the volatility of the manager's promised value by exploiting cross-project diversification. At the same time, the manager gains additional scope for misreporting cash flows through cross-project misreporting strategies. As a result, the optimal contract exhibits three distinct cases. Moreover, in general, λ_i and σ_i no longer enter the optimal contract multiplicatively, in contrast to the one-project benchmark.

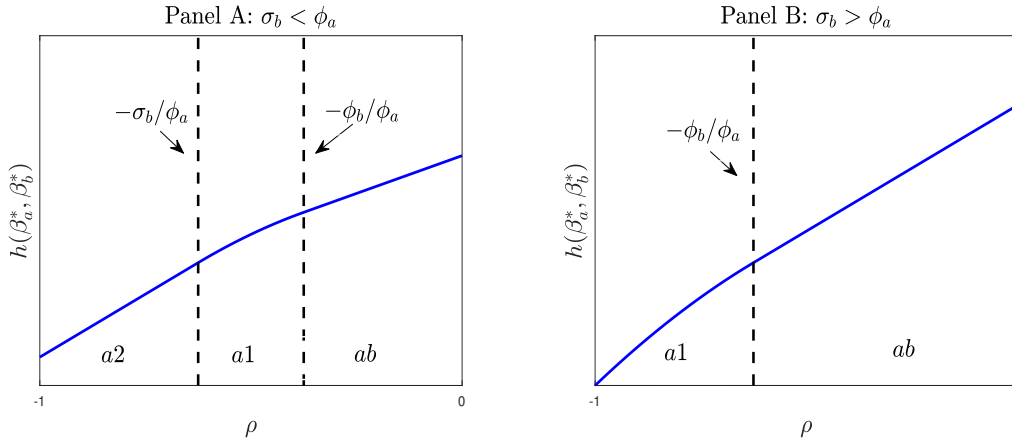


Figure 4: Variance of the manager's promised value as a function of the cash-flow correlation. Each panel plots $h(\beta_a^*, \beta_b^*)$ against ρ ; the dashed lines mark the case boundaries $-\sigma_b/\phi_a$ and $-\phi_b/\phi_a$. In Panel A ($\sigma_b < \phi_a$), all three cases arise as ρ increases: a2, a1, and ab. In Panel B ($\sigma_b > \phi_a$), only cases a1 and ab arise.

Having characterized the conglomerate's optimal contract, we next introduce a measure of risk-adjusted productivity and derive a homogeneity property of the optimal contract, both of which are useful for analyzing the trade-off between centralized and decentralized financing.

¹⁸This is because case a2 would require $-\rho > \sigma_b/\phi_a > 1$, that is, $\rho < -1$, which is infeasible.

3.3.5 Risk-adjusted Productivity η and a Homogeneity Property

For both a stand-alone firm and a conglomerate, it is helpful to introduce the risk-and-agency-adjusted productivity

$$\eta = \frac{\mu}{\phi}, \quad (38)$$

where $\mu > 0$ is the expected cash-flow rate and $\phi > 0$ is the volatility coefficient of the manager's promised value. For a stand-alone project $i \in \{a, b\}$, $\eta_i = \mu_i/\phi_i$ where $\phi_i = \lambda_i \sigma_i$. Because $\phi = \lambda \sigma$ compounds cash-flow volatility with the manager's diversion rate, η adjusts productivity for agency as well as for risk; for brevity we call it risk-adjusted productivity throughout. For a conglomerate, η_c is its expected cash flow scaled by the volatility of its manager's promised value: $\eta_c = \mu_c/\phi_c$ where $\mu_c = \mu_a + \mu_b$ and $\phi_c = \sqrt{h(\beta_a^*, \beta_b^*)}$ given in (27).

Let $P(W; \mu, \eta)$ denote the investor's value function for a firm with expected cash flow μ and risk-adjusted productivity η , so that $P_i(W) = P(W; \mu_i, \eta_i)$ for $i = a, b$ and $P_c(W) = P(W; \mu_c, \eta_c)$.

Proposition 2. *$P(\cdot; \mu, \eta)$ is homogeneous of degree one jointly in (W, μ) , for each fixed η :*

$$P(\theta W; \theta \mu, \eta) = \theta P(W; \mu, \eta), \quad \forall \theta > 0. \quad (39)$$

Proposition 2 is immediate: holding η fixed, rescaling $(W, \mu) \mapsto (\theta W, \theta \mu)$ by a common factor $\theta > 0$ leaves the defining ODE (E.9) in Appendix E and its boundary conditions unchanged up to the same factor θ , so uniqueness of the solution (Lemma 9) gives the homogeneity property.

To illustrate this homogeneity property, consider two firms i and j with the same risk-adjusted productivity: $\eta_i = \eta_j = \eta$. Figure 5 provides an illustration with $\mu_i = 1$ and $\mu_j = 1/2$. Intuitively, when two firms share the same risk-adjusted productivity η but firm i is twice as productive as firm j , firm i is simply a scaled-up version of firm j : under optimal contracting, the manager's promised value in firm i starts twice as high, $W_0^i = 2W_0^j$, and evolves at twice the rate, so W_t stays twice as high and delivers twice the investor's value at every t , almost surely.¹⁹

3.4 Choosing between Centralized and Decentralized Financing

The choice between centralized and decentralized financing boils down to a trade-off between diversification and termination flexibility. Under centralized financing, the manager's

¹⁹Mathematically, as $\mu_i = 1$ and $\mu_j = 1/2$, $P_i(W) = 2P_j(W/2)$ and $P'_i(W) = P'_j(W/2)$.

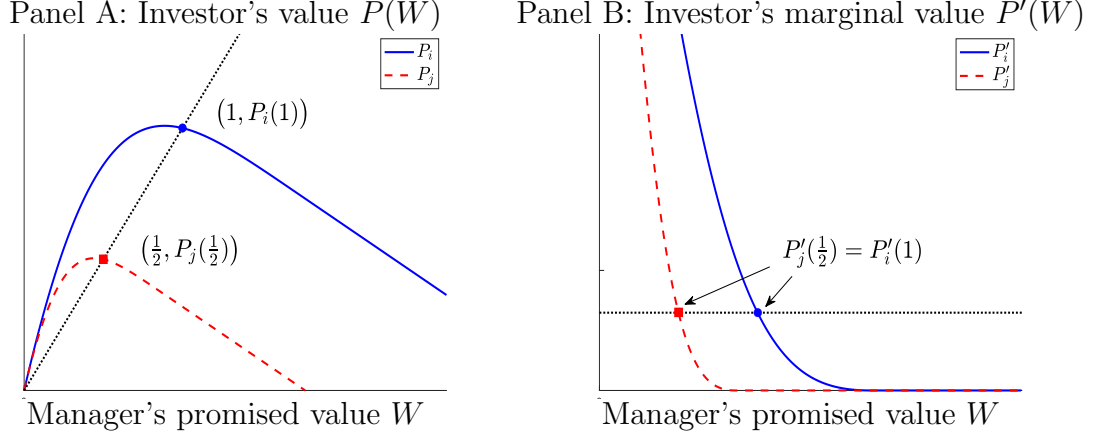


Figure 5: Homogeneity of the investor's value at a common risk-adjusted productivity. Firms i and j have the same η , with $\mu_i = 1$ and $\mu_j = 1/2$. Panel A plots $P_i(W)$ and $P_j(W)$ and illustrates $P_i(W) = 2P_j(W/2)$; Panel B plots $P'_i(W)$ and $P'_j(W)$ and illustrates $P'_i(W) = P'_j(W/2)$. The marked points correspond to $W = 1$ for firm i and $W = 1/2$ for firm j .

promised value loads on both projects' cash-flow shocks, allowing the investor to exploit cross-project diversification to reduce volatility, which is valuable because the investor is averse to inefficient termination. At the same time, bundling ties both projects to a single continuation-value process, so the investor has only one termination option rather than two. Under decentralized financing, each project operates under a separate contract, allowing independent termination. The cost is that each manager's promised value loads only on his own project's shock and cannot benefit from cross-project diversification.

The investor's value is increasing in risk-adjusted productivity, so the comparison collapses to a single cutoff in η_c . Below we formally establish conditions under which the investor chooses between centralized and decentralized financing.

Proposition 3. *Centralized financing is optimal if and only if $\eta_c \geq \hat{\eta}_c$, where the cutoff $\hat{\eta}_c$ is unique and satisfies $\hat{\eta}_c \geq \eta_L := \frac{\mu_a + \mu_b}{\phi_a + \phi_b}$.*

Proposition 3 shows that the organizational choice depends entirely on the conglomerate's risk-adjusted productivity η_c . Centralized financing is strictly preferred only when the volatility reduction from joint contracting raises η_c above $\hat{\eta}_c$, thereby compensating for the loss of independent termination options. Because η_c is strictly decreasing in ρ (see Figure 4) while $\hat{\eta}_c$ does not depend on ρ at all, the two cutoffs are in one-to-one correspondence.²⁰ The map from $\hat{\eta}_c$ to $\hat{\rho}$ is strictly decreasing: the higher the bar that centralization must clear, the more

²⁰The threshold $\hat{\eta}_c$ is pinned down by the two stand-alone values and by $\mu_a + \mu_b$, none of which depends on ρ . The conglomerate's risk-adjusted productivity $\eta_c(\rho) = (\mu_a + \mu_b)/\phi_c(\rho)$ is continuous and strictly decreasing in ρ . There is therefore at most one correlation at which the two sides balance, and we define the cutoff $\hat{\rho} \in [-1, 1]$ by $\eta_c(\hat{\rho}) = \hat{\eta}_c$, setting $\hat{\rho} = 1$ when $\eta_c(1) \geq \hat{\eta}_c$ and $\hat{\rho} = -1$ when $\eta_c(-1) < \hat{\eta}_c$, in which case centralized financing is never optimal.

negative the correlation it needs. Proposition 3 can then be restated in terms of correlation alone: centralized financing is optimal if and only if $\rho \leq \hat{\rho}$. The two corollaries below locate $\hat{\rho}$ in the two polar cases, first when the divisions are equally productive per unit of risk and then when they are not.

Corollary 2. The $\eta_a = \eta_b$ case. *In this case, centralized financing is optimal for every $\rho < 1$ and the investor is indifferent between centralized and decentralized financing if $\rho = 1$.*

At $\rho = 1$ with $\eta_a = \eta_b$, we have $\eta_c = \eta_a = \eta_b$, and Proposition 2 yields $\max_W P_c(W; \rho = 1) = \max_W P_a(W) + \max_W P_b(W)$. Thus, the investor is indifferent at $\rho = 1$. For every $\rho < 1$, joint contracting strictly reduces promised-value volatility and raises the centralized value, so centralized financing is strictly optimal: the cutoff is $\hat{\rho} = 1$.

Corollary 3. The $\eta_a \neq \eta_b$ case. *When the risk-adjusted productivities for the two projects are sufficiently different, decentralized financing is optimal ($\hat{\rho} = -1$).²¹*

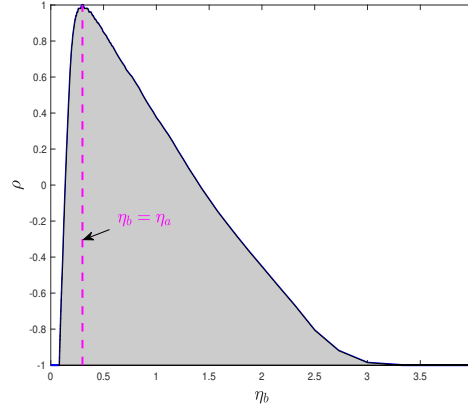


Figure 6: The correlation cutoff $\hat{\rho}$ as a function of η_b . The cutoff is traced by varying σ_b , and hence $\eta_b = \mu_b/(\lambda_b\sigma_b)$, with all other primitives held fixed; along this path both projects' NPVs remain positive. Centralized financing is optimal in the shaded region, below the cutoff. The cutoff peaks at $\hat{\rho} = 1$ where $\eta_b = \eta_a$ (dashed line) and falls to -1 when η_b/η_a is sufficiently small or sufficiently large.

Intuitively speaking, when η_a and η_b are sufficiently different, the project-level independent termination options are so valuable that they dominate the cross-project diversification benefit, even when $\rho = -1$. At the high- η_b end a second force is at work: a very high η_b means a very low σ_b , which puts the conglomerate in case a2 of Subsection 3.3, where the no-overreporting constraint caps the hedge, so project b cannot absorb project a 's volatility however favorable the correlation. Appendix B.6 and B.7 prove the corollaries.

²¹Mathematically, fix r , γ , μ_a , μ_b , λ_a , λ_b , and $\sigma_a > 0$, and vary only $\sigma_b > 0$. There exists a constant $K > 1$, depending on these fixed primitives, such that decentralized financing is optimal ($\hat{\rho} = -1$) whenever $\eta_b/\eta_a < 1/K$ or $\eta_b/\eta_a > K$.

Figure 6 traces the cutoff along exactly the path of Corollary 3, varying η_b through σ_b while holding the remaining non-correlation primitives fixed. The shaded region corresponds to centralized financing; along this path, the cutoff peaks at $\hat{\rho} = 1$ when $\eta_b = \eta_a$ and reaches -1 when η_b/η_a becomes sufficiently small or sufficiently large.

This trade-off between diversification and termination flexibility is fixed once and for all at $t = 0$ in the baseline model.

4 Stochastic Organizational Structure: Conglomerate Inefficiency and Spinoffs

Organizing projects together under a conglomerate can be inefficient—the dark side of internal capital markets (Rajan, Servaes, and Zingales, 2000; Scharfstein and Stein, 2000)—and organizational form is not fixed over time but adjusts as circumstances evolve. We capture these two features by introducing, into the baseline model of Sections 2–3, an operating cost $\kappa > 0$ for the conglomerate and a spinoff option it can exercise at any time. Doing so turns organizational structure into a dynamic state-contingent choice: we characterize when, and which division, a conglomerate spins off, and how this reshapes the $t = 0$ choice between centralized and decentralized financing.

4.1 Formulating the Conglomerate’s Contracting Problem

A conglomerate can spin off either division at any time. Mathematically, it chooses two stopping times, τ_a and τ_b . If $\tau_a < \tau_b$, division a is spun off at τ_a , and the conglomerate is liquidated at τ_b , as division b is the only division remaining before contract termination. Figure 7 illustrates the lifecycle of a conglomerate.

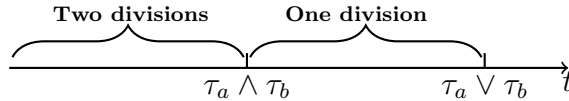


Figure 7: The lifecycle of a conglomerate with a spinoff option. The conglomerate operates two divisions until the first spinoff at $\tau_a \wedge \tau_b$ and one division from then until the contract terminates at $\tau_a \vee \tau_b$.

To capture the inefficiency of operating divisions inside a conglomerate—such as distorted capital allocation or agency conflicts across divisions—we introduce a per-period inefficiency cost $\kappa \geq 0$. This cost reduces the conglomerate-level productivity while leaving each division’s stand-alone productivity unchanged.

Condition 2. *The conglomerate's productivity, μ_c , does not exceed the sum of the two divisions' productivities but exceeds the productivity of either division:*

$$\mu_c := \mu_a + \mu_b - \kappa > \max\{\mu_a, \mu_b\}, \quad (40)$$

where $\kappa \geq 0$ measures the additional cost per unit of time that the investor incurs when operating a conglomerate; $\kappa = 0$ is the benchmark of Section 5.

The restriction $\mu_c > \max\{\mu_a, \mu_b\}$ ensures that operating both divisions jointly remains productive relative to operating either division alone. As in the baseline conglomerate model, for $t < \tau_a \wedge \tau_b$, the manager's continuation-value process is characterized by (15). Unlike in the baseline model, the conglomerate operates as a single-division firm between the spinoff of one division and contract termination, that is, for $\tau_a \wedge \tau_b < t < \tau_a \vee \tau_b$.

Problem 3. *The conglomerate's investor chooses $(\{U_t\}, \tau_a, \tau_b, \beta_a, \beta_b)$ to maximize*

$$F_0 = \mathbb{E} \left[\int_0^{\tau_a \wedge \tau_b} e^{-rt} (\mu_c dt - dU_t) + e^{-r\tau_a} P_b(W_{\tau_a}) \mathbf{1}_{\tau_a \leq \tau_b} + e^{-r\tau_b} P_a(W_{\tau_b}) \mathbf{1}_{\tau_b < \tau_a} \right], \quad (41)$$

subject to the IC conditions (12)–(13) and the manager's continuation-value process given in (15) for $t < \tau := \tau_a \wedge \tau_b$.

Relative to **Problem 1** in the baseline model, **Problem 3** adds two new elements: a real option to spin off either division at stopping times τ_a and τ_b , after which the remaining division continues as a stand-alone firm with value $P_a(\cdot)$ or $P_b(\cdot)$; and the inefficiency cost κ , which reduces the conglomerate's flow productivity from $\mu_a + \mu_b$ to $\mu_c = \mu_a + \mu_b - \kappa$ while both divisions remain integrated. The extended model therefore embeds a dynamic trade-off between the diversification benefit of staying joint and the option value of spinning off to eliminate κ —though κ itself never enters the manager's continuation-value dynamics, since the investor bears the inefficiency directly.

4.2 Solution: Variational Inequality

Let $F(W)$ denote the investor's value function under centralized financing with spinoff options. The extended problem is a combined control and optimal-stopping problem: the investor dynamically chooses incentive intensities (β_a, β_b) while also deciding whether to continue operating the conglomerate, make cash payouts, or spin off a division. Appendix E establishes the viscosity characterization and the contracting properties summarized below.

We characterize $F(W)$ using the following variational inequality:

$$\max \left\{ \underbrace{\mathcal{L}F}_{\substack{\text{continue as} \\ \text{a going concern}}}, \underbrace{-1 - F'}_{\substack{\text{make cash payout} \\ \text{to management}}}, \underbrace{P_a - F}_{\substack{\text{spin off division } b}}, \underbrace{P_b - F}_{\substack{\text{spin off division } a}} \right\} = 0, \quad (42)$$

where

$$\mathcal{L}F := \sup_{(\beta_a, \beta_b) \in \mathcal{C}} \frac{h(\beta_a, \beta_b)}{2} F''(W) + \gamma W F'(W) - rF(W) + \mu_c, \quad (43)$$

subject to the boundary condition at contract termination:

$$F(0) = 0, \quad (44)$$

and the linear-growth condition

$$\lim_{W \rightarrow \infty} \frac{F(W)}{W} = -1. \quad (45)$$

When the first term in (42) equals zero, the two projects are jointly managed under a conglomerate despite the operating inefficiency ($\kappa > 0$). This corresponds to region **I**, in which the investor optimally tracks the manager's promised-value process $\{W_t\}$ given in (15) under the optimal sensitivity pair (β_a^*, β_b^*) , and neither makes cash payments to the manager nor spins off either division.

When the second term in (42) equals zero, the investor optimally makes a cash payment $dU_t = \max\{W_t - \bar{W}, 0\}$, where $\bar{W}$ is the maximum promised value below which the manager receives no cash compensation. This corresponds to the management payout region **P** := $\{W \geq 0 : F'(W) = -1\}$. In this region, the marginal cost of compensating the manager equals one. At states where neither spinoff obstacle binds, the two projects continue to be jointly managed; if a payout and a spinoff obstacle bind simultaneously, the corresponding actions are value-equivalent.

When the third term in (42) equals zero, the investor optimally spins off division b , after which the surviving division a is financed as a stand-alone firm under the optimal contract characterized in DeMarzo and Sannikov (2006). This corresponds to the region **S_b**, defined as **S_b** := $\{W > 0 : F(W) = P_a(W)\}$. In this region, the continuation value of operating both divisions jointly falls below the value of separating and operating division a as a focused firm.

By symmetry, when the fourth term in (42) equals zero, the investor optimally spins off division a and subsequently finances division b as a stand-alone firm. This corresponds to the region **S_a**, defined as **S_a** := $\{W > 0 : F(W) = P_b(W)\}$.

In summary, the investor optimally chooses the policies implied by the variational inequality

ity (42) to maximize his value under centralized financing, $F(W)$. Next, we summarize the structure of the optimal contract under centralized financing in Theorem 2.

Theorem 2. *Problem 3 admits an optimal contract. Its investor value function F is the unique continuous viscosity solution of (42) on $(0, \infty)$ satisfying the boundary condition (44) and the growth condition (45). Moreover, F is concave on $[0, \infty)$.*

There exists a unique payout boundary $\bar{W} > 0$ such that the payout region is $\mathbf{P} = [\bar{W}, \infty)$. At this boundary,

$$F'(\bar{W}) = -1, \quad F''(\bar{W}) = 0, \quad (46)$$

and, for every $W \geq \bar{W}$,

$$F(W) = F(\bar{W}) - (W - \bar{W}). \quad (47)$$

Let (β_a^, β_b^*) denote the unique minimizer of Problem 2. In the interior of the integrated continuation region, this pair is optimal, and within that region it is the unique optimal sensitivity pair wherever $F''(W) < 0$. When $F''(W) = 0$, the sensitivity pair is value-irrelevant; after a spinoff, incentives are governed by the surviving division's stand-alone contract.*

Because spinoff decisions are irreversible, the value function $F(W)$ need not be twice continuously differentiable (C^2) at spinoff boundaries. The value function is C^2 in the interior of each regime and satisfies smooth fit at an interior spinoff boundary, but its second derivative may jump across that boundary. In contrast, at the management cash-compensation boundary $\bar{W}$, payouts occur repeatedly whenever W_t exceeds $\bar{W}$, requiring both smooth pasting and the super-contact condition. Accordingly, $F(W)$ is twice differentiable at $\bar{W}$ and satisfies $F''(\bar{W}) = 0$, as in the baseline model of Section 3.

To ease exposition in the next section, we define the conglomerate's enterprise value as

$$G(W) = F(W) + W. \quad (48)$$

Similarly, let $Q_i(W) = P_i(W) + W$ denote firm i 's enterprise value, where $i = a, b$.

When agency frictions are shut down and $\gamma = r$, organizational form becomes irrelevant for value creation. In this benchmark, incentive distortions vanish, and centralized and decentralized financing generate identical enterprise value:

$$G(W) = \frac{\mu_a + \mu_b}{r} = Q_a(W) + Q_b(W), \quad (49)$$

with $Q_a(W) = \mu_a/r$ and $Q_b(W) = \mu_b/r$. Thus, differences between centralized and decentralized financing arise entirely from the interaction between agency frictions, diversification, and

termination flexibility.

Organizational choice. In the next two sections, we revisit the organizational choice in the extended model with spinoff options. As in Section 2.3, the investor compares the maximal value under centralized financing with the value under decentralized financing. Centralized financing is optimal if

$$\max_W F(W) > P_d^*,$$

and decentralized financing is optimal if $\max_W F(W) < P_d^*$, where P_d^* denotes the investor's maximal value under decentralized financing. The investor is indifferent when equality holds.

5 A Contracting Theory of Corporate Spinoffs: $\kappa = 0$

In this section, we develop a theory of spinoffs for the $\kappa = 0$ case, as it allows us to isolate the pure contracting-theoretic force driving spinoffs.²² A key takeaway is that if the conglomerate spins off a division, it is the one with the lower risk-adjusted productivity η , and it does so when the manager's promised value falls below an endogenous threshold. Existing theories attribute spinoffs to information asymmetry between managers and outside investors (Krishnaswami and Subramaniam, 1999) or to financing needs and firm scope (Nanda and Narayanan, 1999); see Maksimovic and Phillips (2013) for a survey of the evidence. Here the spinoff decision is derived from the optimal contract itself: it is a state-contingent trigger tied to the manager's promised value, with no informational friction between the firm and outside investors. Next, we state the $t = 0$ organizational choice.

Proposition 4. The $\kappa = 0$ case. *At $t = 0$, centralized financing is always (weakly) optimal.*

Appendix C.1 provides a proof. To see the result, consider perfectly correlated projects, $\rho = 1$, which is least favorable to centralization.²³ With a single shock driving both projects, the conglomerate manager's promised value reveals the shock path, and hence what each stand-alone contract would have promised its manager. One contract with the conglomerate's manager can therefore span the two stand-alone contracts: the investor pays out whatever each stand-alone contract would pay, and spins off a division at the moment its stand-alone contract would liquidate it. This replicates the decentralized value, so centralized financing weakly dominates even in the least favorable case. The spinoff option is essential here. Without it, the conglomerate cannot terminate one division while keeping the other, and Proposition 3 shows that decentralized financing can then be strictly better.

²²Section 6 extends it, qualitatively unchanged, to $\kappa > 0$.

²³Lowering ρ increases diversification and lowers the volatility of the manager's promised value. Because the investor's value is concave, lower volatility raises it. The conglomerate is therefore worth at least as much at any $\rho < 1$ as at $\rho = 1$, and hence at least the decentralized value.

Next, we characterize which division is spun off in Proposition 5 and the optimal spinoff cutoff rule in Proposition 6.

Proposition 5. Optimal spinoff: the $\kappa = 0$ case. *If $\eta_a \neq \eta_b$ and $\eta_c < \max\{\eta_a, \eta_b\}$, the conglomerate always spins off the lower- η division first, strictly before the remaining division is liquidated: $\tau_a \wedge \tau_b < \tau_a \vee \tau_b$, almost surely. Otherwise, the conglomerate is liquidated as a whole: $\tau_c = \tau_a = \tau_b$.*

Proposition 5 pins down the timing of the spinoff. The logic is survival. Near termination the investor's value is governed by risk-adjusted productivity η , so what matters is whether the conglomerate or one of its divisions on its own offers the higher η . Consider the two cases in turn. In the first case, $\eta_a \neq \eta_b$ and $\eta_c < \max\{\eta_a, \eta_b\}$, retaining the higher- η division alone raises risk-adjusted productivity from η_c to $\max\{\eta_a, \eta_b\}$. The conglomerate therefore spins off the lower- η division, and it does so strictly before liquidation. In the second case, $\eta_c \geq \max\{\eta_a, \eta_b\}$, diversification makes the conglomerate at least as productive per unit of risk as either division on its own, so spinning off either division would lower risk-adjusted productivity. This implies that the option value of separately spinning off a division is zero and the conglomerate's liquidation is the only economically meaningful real option. The symmetric case $\eta_a = \eta_b$ belongs here, because diversification implies $\eta_c \geq \eta_L = \eta_a = \eta_b$.

Next, Proposition 6 characterizes the optimal spinoff threshold $\widehat{W}$ for the first case where spinoff is valuable.

Proposition 6. The spinoff boundary: the $\kappa = 0$ case. *If $\eta_a \neq \eta_b$ and $\eta_c < \max\{\eta_a, \eta_b\}$, there exists a unique cutoff $\widehat{W} \in (0, \overline{W})$ below which the conglomerate spins off the lower- η division. Without loss of generality, suppose $\eta_b > \eta_a$. Then the conglomerate spins off division a the first time the manager's promised value falls to $\widehat{W}$: $\tau_a = \inf\{t \geq 0 : W_t \leq \widehat{W}\}$. The cutoff $\widehat{W}$ solves the following value-matching and smooth-pasting conditions:*

$$F(\widehat{W}) = P_b(\widehat{W}), \quad F'(\widehat{W}) = P'_b(\widehat{W}). \quad (50)$$

See Appendices C.2 and C.3 for proofs of Propositions 5 and 6.

Figure 8 illustrates the solution for the case $\kappa = 0$. We plot the conglomerate's enterprise value $G(W) = F(W) + W$, where $F(W)$ is the investor's value characterized in Theorem 2. Suppose the risk-adjusted productivity of division b exceeds that of division a , i.e., $\eta_b > \eta_a$. The figure exhibits three distinct regions.

In the left region, where W is low and contract termination becomes likely, the conglomerate optimally spins off the lower- η division a in order to preserve the higher- η division b and increase survival probability. In the middle region, the conglomerate operates as a going concern and pays no cash compensation. In the right region, the conglomerate is financially

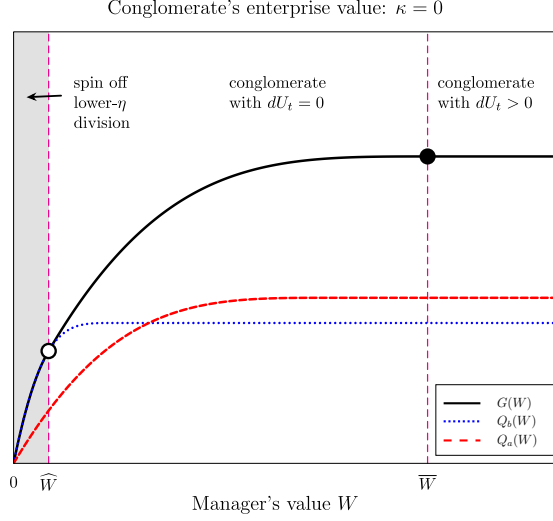


Figure 8: The conglomerate's enterprise value under $\kappa = 0$, with $\eta_b > \eta_a$. The solid line is the conglomerate's enterprise value $G(W)$; the dotted and dashed lines are the stand-alone enterprise values $Q_b(W)$ and $Q_a(W)$. Below the spinoff cutoff $\widehat{W}$ (open circle), the conglomerate spins off the lower- η division a ; above the payout cutoff $\overline{W}$ (filled circle), it pays out cash.

unconstrained in that $G'(W) = 0$ and therefore pays a lump-sum cash compensation to the manager. Note that $Q_a(W) > Q_b(W)$ for sufficiently large W , as $\mu_a > \mu_b$ in this example.

The two cutoffs in Figure 8, the spinoff threshold $\widehat{W}$ and the payout boundary $\overline{W}$, play different roles. Economically, $\widehat{W}$ marks where the diversification benefit of staying joint stops offsetting the termination-risk cost of doing so, while $\overline{W}$ marks where deferring the manager's compensation becomes as costly as paying it out. They also differ in the smoothness they require of F . The figure marks this distinction with an open circle at $\widehat{W}$ and a filled circle at $\overline{W}$.²⁴

In summary, with $\kappa = 0$, a single criterion, risk-adjusted productivity η , fully characterizes the spinoff decision. We next reintroduce the dark side of internal capital markets, $\kappa > 0$, and show that this single addition fundamentally changes the analysis.

6 The Dark Side of Internal Capital Markets: Spinoff Reversal and the Limits of Centralization

Positive κ breaks the $\kappa = 0$ benchmark's clean logic: staying joint is no longer costless, so spinning off now does double duty—cutting κ as well as managing termination risk. Level productivity μ therefore joins risk-adjusted productivity η as a second, possibly conflicting,

²⁴ $\widehat{W}$ is the exercise point of an irreversible option, so it needs only value matching and smooth pasting, and F is merely C^1 there. $\overline{W}$, by contrast, is reached repeatedly—cash is paid every time W_t exceeds it—and so additionally requires the super-contact condition $F''(\overline{W}) = 0$.

criterion for which division to keep. The same cost puts a genuine price on centralization at $t = 0$: it dominates only when diversification's gain in η outweighs κ .

6.1 Spinoff Reversal: Two Separate Episodes

Consistent with Condition 1, let $\eta_b > \eta_a$ and consider the case $\mu_b < \mu_a$, where the two rankings conflict. These inequalities imply $\phi_b = \mu_b/\eta_b < \mu_a/\eta_a = \phi_a$, preserving the maintained labeling $\phi_a \geq \phi_b$. At small W , termination risk is high and survival dominates: division b 's higher η makes retaining it more valuable, so $Q_b(W) > Q_a(W)$. At large W , termination risk is negligible and level productivity μ dominates instead: division a 's higher μ makes retaining it more valuable, so $Q_b(W) < Q_a(W)$ —the spun-off division's identity reverses as the manager's promised value evolves.

These two stand-alone values cross at most once (Lemma 7 in Appendix D.1), and the crossing organizes the optimal policy: the conglomerate sheds division a only below it and division b only above it. When each division is shed on an interval of promised values, the optimal policy has five regions. Next, we state this result for the general case in which both spinoff regions have nonempty interiors.²⁵

Proposition 7. Five policy regions: the $\kappa > 0$ case. *Suppose $\eta_b > \eta_a$ and $\mu_b < \mu_a$, and let $\mu_c = \mu_a + \mu_b - \kappa > \max\{\mu_a, \mu_b\}$. Then the optimal policy consists of five connected regions, with boundaries $0 < \widehat{W}_1 < \widehat{W}_2 < \widehat{W}_3 < \overline{W}$, ordered as*

$$\underbrace{\mathbf{S}_a = (0, \widehat{W}_1]}_{\text{spin off } a}, \quad \underbrace{\mathbf{I}_1 = (\widehat{W}_1, \widehat{W}_2)}_{\text{keep both}}, \quad \underbrace{\mathbf{S}_b = [\widehat{W}_2, \widehat{W}_3]}_{\text{spin off } b}, \quad \underbrace{\mathbf{I}_2 = (\widehat{W}_3, \overline{W})}_{\text{keep both}}, \quad \underbrace{\mathbf{P} = [\overline{W}, \infty)}_{\text{pay out}}.$$

The conglomerate spins off division a on $\mathbf{S}_a$ and division b on $\mathbf{S}_b$, keeps both divisions without payout on $\mathbf{I}_1$ and $\mathbf{I}_2$, and pays out cash on $\mathbf{P}$. The investor's value solves $\mathcal{L}F = 0$ on $\mathbf{I}_1$ and $\mathbf{I}_2$, with $F(0) = 0$, and satisfies value matching and smooth pasting at the three spinoff boundaries:

$$\begin{aligned} F(\widehat{W}_1) &= P_b(\widehat{W}_1), & F'(\widehat{W}_1) &= P'_b(\widehat{W}_1); \\ F(\widehat{W}_j) &= P_a(\widehat{W}_j), & F'(\widehat{W}_j) &= P'_a(\widehat{W}_j), \quad j = 2, 3. \end{aligned} \tag{51}$$

At the payout boundary,

$$F'(\overline{W}) = -1 \quad \text{and} \quad F''(\overline{W}) = 0, \tag{52}$$

²⁵Appendix D.1 provides a proof. Corollary 7 in Appendix D.2 covers the configurations with fewer than five regions.

as in Theorem 2.

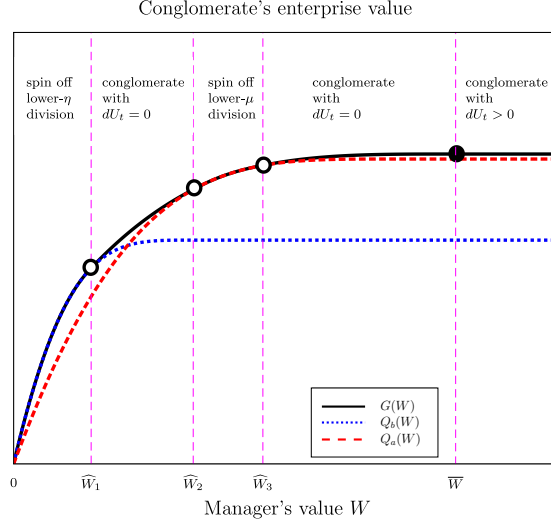


Figure 9: The conglomerate's enterprise value under $\kappa > 0$, with $\eta_b > \eta_a$ and $\mu_b < \mu_a$: the five-region case. The solid line is $G(W)$; the dotted and dashed lines are $Q_b(W)$ and $Q_a(W)$. In this example the two stand-alone values cross once and both spinoff sets have nonempty interiors, which produces the maximal five-region configuration (Proposition 7). The conglomerate spins off the lower- η division a for $W \leq \widehat{W}_1$ and the lower- μ division b for $\widehat{W}_2 \leq W \leq \widehat{W}_3$; elsewhere it keeps both divisions, paying out cash once W reaches $\overline{W}$.

Figure 9 illustrates Proposition 7. It plots the conglomerate's enterprise value $G(W)$ under the optimal contract, which divides into five regions.

Region 1 (Low W): Spin off lower- η division a . Low promised value W means high termination risk, so survival dominates. Spinning off division a raises risk-adjusted productivity from η_c to $\eta_b > \eta_a$ —improving firm value and lowering termination risk—while also eliminating the inefficiency cost κ .

Region 2 (Medium-Low W): Retain both divisions, no cash payout to manager. Termination risk is still relevant, but the option value of keeping the conglomerate as a going concern dominates immediate separation: the conglomerate retains both divisions and adjusts promised value through performance-based incentives.

Region 3 (Middle W): Spin off lower- μ division b . Here both survival and level productivity matter. The investor keeps the higher-productivity division a and spins off b , which raises risk-adjusted productivity from η_c to η_a provided κ is large enough: $\eta_c = \frac{\mu_a + \mu_b - \kappa}{\phi_c} < \eta_a < \eta_b$. Separation therefore improves both operational efficiency and risk-adjusted performance.

Region 4 (Medium-High W): Retain both divisions, no cash payout to manager. Termination risk is now moderate: the conglomerate keeps both divisions for their diversification and

option value, but marginal promised value still exceeds one, so cash compensation is not yet optimal.

Region 5 (High W): Retain both divisions and pay cash to manager. Once W exceeds the payout boundary $\overline{W}$, the conglomerate is financially unconstrained and pays lump-sum cash compensation $dU_t = W_t - \overline{W}$, continuing joint operation despite $\kappa > 0$ because diversification still dominates at high continuation values.

In sum, the optimal contract is non-monotonic in W : survival dominates at low W (spin off the lower- η division), productivity dominates at intermediate W (spin off the lower- μ division instead), and at high W the conglomerate stays intact, eventually paying cash. Spinoff therefore happens in two disjoint episodes, not one continuous interval, separated by a region where the conglomerate remains whole—reflecting the interaction of risk-adjusted productivity η , level productivity μ , and the operating cost κ .²⁶

Comparison with liquidity-driven divestiture. In Dai et al. (2024), costly external finance makes a division sale a source of funds, so divestiture occurs only when liquidity is scarce, and it is the smaller division that goes. Here the spinoff hazard is positive in distress, zero over an intermediate range, and positive again after recovery, with a different division shed each time. Divestitures by healthy firms, and a switch in which division the same firm sheds, are consistent with our theory but not with a liquidity-driven one.

Appendix D.3 records an additional result for the case in which the two rankings agree instead of conflict: there the tension behind the five regions disappears, and a single criterion again governs the spinoff decision.

6.2 Choosing between Centralized and Decentralized Financing

Once centralization is no longer costless, the organizational choice trades off two forces: centralization imposes an operating inefficiency $\kappa > 0$ (the dark side of internal capital markets) but may lower the volatility of the manager’s promised value W through diversification, raising risk-adjusted productivity η_c (the bright side).

To formalize this trade-off, first define the following object:

$$\hat{\kappa} := \mu_a + \mu_b - rP_d^*, \quad (53)$$

where $P_d^* = \max_{W \geq 0} P_a(W) + \max_{W \geq 0} P_b(W)$ is the investor’s total value under decentralized financing. Since the first-best flow payoff is $\mu_a + \mu_b$ absent agency frictions, $\hat{\kappa}$ defined in (53)

²⁶General Electric illustrates the pattern: it sold its biopharma business in 2019 to pay down debt and, once its balance sheet had recovered, split into three focused companies over 2023–2024.

is the resulting productivity loss—the wedge between the first-best output and the investor’s annualized value under decentralization: rP_d^* .

Comparing κ to $\hat{\kappa}$ settles the organizational choice. Below we summarize the results.

Proposition 8. (i) If $\kappa \geq \hat{\kappa}$, decentralized financing at $t = 0$ is always optimal; (ii) If $0 < \kappa < \hat{\kappa}$, there exists a unique cutoff $\hat{\eta}_c(\kappa)$ such that centralized financing at $t = 0$ is optimal if and only if $\eta_c \geq \hat{\eta}_c(\kappa)$. The cutoff $\hat{\eta}_c(\kappa)$ is increasing in κ and diverges as κ approaches $\hat{\kappa}$.

Intuitively, $\kappa \geq \hat{\kappa}$ means that the dark side of internal capital markets alone outweighs any bright-side benefit, so decentralization at $t = 0$ always wins. For $\kappa < \hat{\kappa}$, the cutoff $\hat{\eta}_c(\kappa)$ marks exactly the point at which the diversification gain in risk-adjusted productivity η_c offsets the dark-side cost κ . This cutoff is the counterpart, in the extended model, of the baseline cutoff $\hat{\eta}_c$ of Proposition 3. At $\kappa = 0$, centralized financing is always optimal (Proposition 4), so the cutoff equals η_L , the lowest value that η_c can take. A positive κ raises the bar, and the cutoff grows without bound as κ approaches $\hat{\kappa}$. Finally, once a conglomerate is formed at $t = 0$, its organizational structure continues to evolve: the firm moves across the five regions of Proposition 7, illustrated in Figure 9.

When the two divisions have the same risk-adjusted productivity, their ranking by expected cash flow alone determines which division can be spun off. The following corollary gives the corresponding condition on the running cost and cash-flow correlation.

Corollary 4. Spinoffs with equal divisional risk-adjusted productivity: $\eta_a = \eta_b > 0$. Without loss of generality, let $\mu_a > \mu_b$. Under Condition 2, division a strictly dominates division b at all times: $P_a(W) > P_b(W)$ for every $W > 0$. Consequently, the investor never spins off division a : $\mathbf{S}_a = \emptyset$. Therefore, there are at most three regions:

$$\underbrace{\mathbf{S}_b = (0, \widehat{W}]}_{\text{spin off } b}, \quad \underbrace{\mathbf{I} = (\widehat{W}, \overline{W})}_{\text{keep both}}, \quad \text{and} \quad \underbrace{\mathbf{P} = [\overline{W}, \infty)}_{\text{pay out}}.$$

The latter two regions, $\mathbf{I}$ and $\mathbf{P}$, always exist, and $\overline{W}$ is characterized by the same smooth-pasting and super-contact conditions (52) as in Proposition 7. Finally,

(i) the spinoff region exists, $\mathbf{S}_b \neq \emptyset$, if and only if $\eta_c < \eta_a = \eta_b$, or equivalently $\kappa > \mu_a + \mu_b - \sqrt{\mu_a^2 + \mu_b^2 + 2\rho\mu_a\mu_b}$. The spinoff boundary $\widehat{W}$ is then characterized by the value-matching and smooth-pasting conditions (51) as in Proposition 7, namely $F(\widehat{W}) = P_a(\widehat{W})$ and $F'(\widehat{W}) = P'_a(\widehat{W})$.

(ii) Otherwise, i.e., if $\eta_c \geq \eta_a = \eta_b$, there is no spinoff region: $\widehat{W} = 0$.

Appendix D.5 provides a proof of Corollary 4.

7 Conclusion

This paper develops a theory of internal capital markets and corporate spinoffs. An investor finances two projects whose cash flows a manager privately observes and can divert. The optimal contract governs the manager's compensation and the firm's organizational form: whether the two divisions are financed under one roof at inception, and whether, when, and which one is later spun off.

Financing two correlated divisions jointly lowers the volatility of the manager's promised value, delivering incentive-compatible coinsurance. Under the optimal contract, the no-underreporting constraint always binds for the division with the higher agency measure but need not bind for the other, whose cash flow can serve as a hedge. The benefit comes at the cost of bundled survival: liquidation, when it comes, claims both divisions. Whether to centralize at inception turns on one statistic, the conglomerate's risk-adjusted productivity.

A spinoff option loosens that bundle and makes breakup state-contingent. Absent any cost of operating a conglomerate ($\kappa = 0$), the firm breaks up only under financial distress, shedding the division with lower risk-adjusted productivity to survive. With such a cost ($\kappa > 0$), a second motive appears: when financially sound, the firm spins off the less productive division to refocus. The two motives generate two disjoint spinoff regions, one for risk management and the other for efficiency.

The bright and dark sides of internal capital markets therefore need not be properties of different firms. They can coexist within one conglomerate, and which one dominates is then settled continuously by the firm's distance to distress rather than once by its primitives. The model's sharpest prediction is this reversal: which division a conglomerate sheds depends on its financial condition at the time. That prediction awaits empirical testing.

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Appendices

A Extensions of the Model in Section 2

A.1 Outside Savings Option

We now introduce a private outside saving account for the manager. This account earns a return $r_o \leq r$ and is unobservable to the investor. The manager may withdraw funds for consumption or secretly divert firm cash flows into this account. Let S_t denote the account balance. Its dynamics are

$$dS_t = r_o S_t dt + [dY_t^a - d\hat{Y}_t^a]^{\lambda_a} + [dY_t^b - d\hat{Y}_t^b]^{\lambda_b} + dU_t - dC_t, \quad S_0 = 0, \quad (\text{A.1})$$

where C_t is any nondecreasing consumption process such that $S_t \geq 0$ for all $t \geq 0$.

The next proposition shows that private saving cannot increase the manager's utility and that a no-saving strategy remains optimal.

Proposition 9. *Under the baseline optimal contract, whose sensitivities satisfy $\lambda_i \leq \beta_i^* \leq 1$ for $i \in \{a, b\}$, define*

$$\hat{V}_t := \int_0^t e^{-\gamma s} dC_s + e^{-\gamma t} (S_t + W_t). \quad (\text{A.2})$$

Then the stopped process $\{\hat{V}_{t \wedge \tau}\}_{t \geq 0}$ is a supermartingale.

This implies

$$W_0 = \hat{V}_0 \geq \mathbb{E}[\hat{V}_{\tau \wedge T}] \quad \text{for all } T > 0. \quad (\text{A.3})$$

By Fatou's lemma,

$$W_0 \geq \limsup_{T \rightarrow \infty} \mathbb{E}[\hat{V}_{\tau \wedge T}] \geq \mathbb{E}[\hat{V}_\tau] = \mathbb{E}\left[\int_0^\tau e^{-\gamma s} dC_s + e^{-\gamma \tau} S_\tau\right] \geq \mathbb{E}\left[\int_0^\tau e^{-\gamma s} dC_s\right]. \quad (\text{A.4})$$

Private saving cannot benefit the manager, and truthful reporting with $S_t = 0$ and $C_t = U_t$ remains optimal. Hence allowing hidden saving does not change the baseline optimal contract or investor value.

Proof of Proposition 9.

$$d\hat{V}_t = e^{-\gamma t} dC_t - \gamma e^{-\gamma t} (S_t + W_t) dt + e^{-\gamma t} (dS_t + dW_t) \quad (\text{A.5})$$

$$\begin{aligned} &= e^{-\gamma t} \left(dC_t - \gamma (S_t + W_t) dt + r_o S_t dt + [dY_t^a - d\hat{Y}_t^a]^{\lambda_a} + [dY_t^b - d\hat{Y}_t^b]^{\lambda_b} + dU_t - dC_t \right. \\ &\quad \left. + \gamma W_t dt - dU_t - \beta_a^* (dY_t^a - d\hat{Y}_t^a) - \beta_b^* (dY_t^b - d\hat{Y}_t^b) + \beta_a^* \sigma_a dZ_t^a + \beta_b^* \sigma_b dZ_t^b \right) \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} &= -e^{-\gamma t} \left((\gamma - r_o) S_t dt + (\beta_a^* - \lambda_a) (dY_t^a - d\hat{Y}_t^a)^+ + (1 - \beta_a^*) (d\hat{Y}_t^a - dY_t^a)^+ \right. \\ &\quad \left. + (\beta_b^* - \lambda_b) (dY_t^b - d\hat{Y}_t^b)^+ + (1 - \beta_b^*) (d\hat{Y}_t^b - dY_t^b)^+ - \beta_a^* \sigma_a dZ_t^a - \beta_b^* \sigma_b dZ_t^b \right). \end{aligned} \quad (\text{A.7})$$

Since $r_o \leq r < \gamma$ and $\lambda_i \leq \beta_i^* \leq 1$ for $i \in \{a, b\}$, the finite-variation part of $\hat{V}_{t \wedge \tau}$ is nonpositive. Subject to the usual integrability condition for the stochastic integral, $\hat{V}_{t \wedge \tau}$ is therefore a supermartingale. $\square$

A.2 General Liquidation Value and Outside Option Value

We generalize the model as follows. Following DeMarzo and Sannikov (2006), we assume that at the time the contract is terminated, the manager receives $R_i \geq 0$ dollars as an outside option payoff, and the investor receives L_i dollars as a liquidation payoff, $i = a, b, c$. For each $i \in \{a, b, c\}$, $P_i : [R_i, \infty) \rightarrow \mathbb{R}$ satisfies

$$\max \left\{ \frac{\phi_i^2}{2} P_i''(W) + \gamma W P_i'(W) + \mu_i - r P_i(W), -P_i'(W) - 1 \right\} = 0, \quad P_i(R_i) = L_i, \quad i = a, b, c \quad (\text{A.8})$$

together with the growth condition

$$\lim_{W \rightarrow \infty} \frac{P_i(W)}{W} = -1. \quad (\text{A.9})$$

For the conglomerate firm with a spinoff option, suppose $R_c \geq \max\{R_a, R_b\}$ so that both stand-alone continuation values are defined throughout the conglomerate's state space. The value function $F : [R_c, \infty) \rightarrow \mathbb{R}$ satisfies the following variational inequality:

$$\begin{aligned} &\max \{ \mathcal{L}F, -1 - F', P_a + L_b - F, P_b + L_a - F \} = 0, \\ &F(R_c) = \max \{ L_c, P_a(R_c) + L_b, P_b(R_c) + L_a \}. \end{aligned} \quad (\text{A.10})$$

together with the growth condition

$$\lim_{W \rightarrow \infty} \frac{F(W)}{W} = -1. \quad (\text{A.11})$$

At $W = R_c$, the investor may either liquidate the conglomerate or exercise either spinoff option. Under promise rollover, the manager enters the surviving firm with promised utility R_c .

B Proofs for Section 3

B.1 Martingale Representation of the Promised Value

Assume

$$dY_t^i - d\hat{Y}_t^i = \xi_t^i dt, \quad i = a, b, \quad (\text{B.1})$$

where ξ_t^i , $i = a, b$, are bounded progressively measurable processes. Set

$$v_t^i := \frac{\xi_t^i}{\sigma_i}, \quad \mathbf{v}_t := \begin{pmatrix} v_t^a \\ v_t^b \end{pmatrix}, \quad C_\rho := \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}. \quad (\text{B.2})$$

Writing $\mathbf{Z}_t := (Z_t^a, Z_t^b)^\top$ and $\hat{\mathbf{Z}}_t := (\hat{Z}_t^a, \hat{Z}_t^b)^\top$, the reporting rule implies

$$d\hat{\mathbf{Z}}_t = d\mathbf{Z}_t - \mathbf{v}_t dt. \quad (\text{B.3})$$

Because $|\rho| < 1$, the covariance matrix C_ρ is invertible. Boundedness of $\mathbf{v}$ implies Novikov's condition on every finite horizon. Define $\hat{\mathbb{P}}$ by

$$\left. \frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \exp \left\{ \int_0^t (C_\rho^{-1} \mathbf{v}_s)^\top d\mathbf{Z}_s - \frac{1}{2} \int_0^t \mathbf{v}_s^\top C_\rho^{-1} \mathbf{v}_s ds \right\}. \quad (\text{B.4})$$

Equivalently, since $C_\rho^{-1} = (1 - \rho^2)^{-1} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$, the density can be written componentwise as

$$\begin{aligned} \left. \frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \exp \left\{ \frac{1}{1 - \rho^2} \int_0^t (v_s^a - \rho v_s^b) dZ_s^a + \frac{1}{1 - \rho^2} \int_0^t (v_s^b - \rho v_s^a) dZ_s^b \right. \\ \left. - \frac{1}{2(1 - \rho^2)} \int_0^t [(v_s^a)^2 - 2\rho v_s^a v_s^b + (v_s^b)^2] ds \right\}. \end{aligned}$$

Indeed, Girsanov's theorem implies that

$$d\mathbf{Z}_t - C_\rho(C_\rho^{-1}\mathbf{v}_t) dt = d\mathbf{Z}_t - \mathbf{v}_t dt = d\widehat{\mathbf{Z}}_t \quad (\text{B.5})$$

is a bivariate Brownian increment under $\widehat{\mathbb{P}}$, with covariance matrix C_ρ .

Assume that the contract satisfies the square-integrability condition

$$\widehat{\mathbb{E}} \left[\left(\int_{[0,\tau]} e^{-\gamma s} dU_s \right)^2 \right] < \infty.$$

We take $U_t = U_\tau$ and $W_t = 0$ for $t \geq \tau$. We also use the ex-payment convention: the integral over $[0, t \wedge \tau]$ includes any transfer at $t \wedge \tau$, whereas $W_{t \wedge \tau}$ contains only transfers strictly after $t \wedge \tau$. Thus, a jump in U is counted exactly once.

Following DeMarzo and Sannikov (2006), we define the stopped process

$$V_{t \wedge \tau} := e^{-\gamma(t \wedge \tau)} W_{t \wedge \tau}(\widehat{\mathbf{Y}}^{t \wedge \tau}) + \int_{[0, t \wedge \tau]} e^{-\gamma s} dU_s. \quad (\text{B.6})$$

By the definition of continuation utility and the convention above,

$$V_{t \wedge \tau} = \widehat{\mathbb{E}} \left[\int_{[0,\tau]} e^{-\gamma s} dU_s \middle| \widehat{\mathcal{F}}_{t \wedge \tau} \right].$$

Consequently, $\{V_{t \wedge \tau}\}$ is a square-integrable $(\widehat{\mathcal{F}}_t, \widehat{\mathbb{P}})$ -martingale, where $\widehat{\mathbb{P}}$ is the probability measure constructed above. Therefore, by the martingale representation theorem, there exist $\widehat{\mathcal{F}}_t$ -predictable, square-integrable processes $\zeta_{a,t}$ and $\zeta_{b,t}$ such that, for $t < \tau$,

$$dV_t = \zeta_{a,t}(d\widehat{Y}_t^a - \mu_a dt) + \zeta_{b,t}(d\widehat{Y}_t^b - \mu_b dt). \quad (\text{B.7})$$

On the other hand, for $t < \tau$, differentiating V_t directly yields

$$dV_t = -\gamma e^{-\gamma t} W_t(\widehat{\mathbf{Y}}^t) dt + e^{-\gamma t} dW_t(\widehat{\mathbf{Y}}^t) + e^{-\gamma t} dU_t. \quad (\text{B.8})$$

Matching the diffusion and drift terms in the two expressions above yields the promised-utility dynamics (11).

B.2 Proof of Corollary 1

Since $P_c''(W) < 0$ in the HJB equation (25), maximizing its right-hand side is equivalent

to minimizing $h(\beta_a, \beta_b)$, defined in (26):

$$\min_{(\beta_a, \beta_b) \in \mathcal{C}} h(\beta_a, \beta_b). \quad (\text{B.9})$$

It is helpful to introduce the following notation:

$$\zeta_i = \beta_i \sigma_i, \quad i = a, b. \quad (\text{B.10})$$

Mathematically, (B.9) is equivalent to the following problem:

$$\text{Minimize } H(\zeta_a, \zeta_b) := \zeta_a^2 + 2\rho \zeta_a \zeta_b + \zeta_b^2, \quad \text{subject to } \zeta_i \geq \phi_i, \quad \frac{\lambda_b \sigma_b}{\sigma_a} \leq \frac{\zeta_b}{\zeta_a} \leq \frac{\sigma_b}{\lambda_a \sigma_a}. \quad (\text{B.11})$$

Note that for any $C \geq 0$,

$$H(C\zeta_a, C\zeta_b) = C^2 H(\zeta_a, \zeta_b), \quad (\text{B.12})$$

so at least one of the constraints $\zeta_i \geq \phi_i$ must bind at the optimum. Otherwise, if both inequalities were slack, we could choose $0 < C < 1$ to scale down (ζ_a, ζ_b) while still satisfying $\zeta_i \geq \phi_i$, and the ratio between ζ_a and ζ_b would remain unchanged, contradicting optimality.

Under Condition 1, $\phi_a \geq \phi_b$. Consider a candidate optimum where

$$\zeta_b = \phi_b, \quad \zeta_a > \phi_a. \quad (\text{B.13})$$

At this point,

$$\partial_{\zeta_a} H(\zeta_a, \zeta_b) = 2(\zeta_a + \rho \zeta_b) = 2(\zeta_a + \rho \phi_b). \quad (\text{B.14})$$

Since $\zeta_a > \phi_a \geq \phi_b$ and $\rho \in [-1, 1]$, we have

$$\partial_{\zeta_a} H(\zeta_a, \phi_b) = 2(\zeta_a + \rho \phi_b) \geq 2(\zeta_a - \phi_b) > 0. \quad (\text{B.15})$$

Thus, decreasing ζ_a strictly reduces H while still maintaining $\zeta_a \geq \phi_a$, contradicting the assumption that $\zeta_a > \phi_a$ is optimal.

Therefore, the alternative $\zeta_a > \phi_a$ is impossible. Since $\zeta_a \geq \phi_a$ by (B.11), we must have $\zeta_a^* = \phi_a = \lambda_a \sigma_a$, and hence $\beta_a^* = \lambda_a$, proving Corollary 1.

B.3 Proof of Proposition 2

By Lemma 9 and Theorem 1, every firm $\ell \in \{a, b, c\}$ has the same reduced contracting problem with parameters (μ_ℓ, ϕ_ℓ) :

$$dW_t^\ell = \gamma W_t^\ell dt - dU_t^\ell + \phi_\ell d\mathcal{B}_t^\ell, \quad \tau_\ell := \inf\{t \geq 0 : W_t^\ell = 0\}. \quad (\text{B.16})$$

For a stand-alone firm, $\mathcal{B}^\ell = \mathcal{Z}^\ell$. For the conglomerate, Theorem 1 permits us to define the standard Brownian motion

$$d\mathcal{B}_t^c := \frac{\beta_a^* \sigma_a d\mathcal{Z}_t^a + \beta_b^* \sigma_b d\mathcal{Z}_t^b}{\phi_c}, \quad \phi_c^2 = h(\beta_a^*, \beta_b^*). \quad (\text{B.17})$$

Fix $i, j \in \{a, b, c\}$ such that $\eta_i = \eta_j$, and define

$$q := \frac{\mu_j}{\mu_i} > 0. \quad (\text{B.18})$$

Because $\eta_\ell = \mu_\ell / \phi_\ell$, equality of risk-adjusted productivities implies

$$q = \frac{\mu_j}{\mu_i} = \frac{\phi_j}{\phi_i}. \quad (\text{B.19})$$

Fix any $w \geq 0$ and consider firm i under an optimal policy, initialized at $W_0^i = \mu_i w$. Let $\bar{W}_i$, U^i , and τ_i denote its optimal payout boundary, cumulative payout process, and liquidation time. On a common Brownian probability space, define a candidate policy for firm j by

$$\widetilde{W}_t^j := qW_t^i, \quad \widetilde{U}_t^j := qU_t^i, \quad \widetilde{\bar{W}}_j := q\bar{W}_i, \quad \widetilde{\tau}_j := \tau_i. \quad (\text{B.20})$$

Then $\widetilde{W}_0^j = \mu_j w$, and, using (B.19),

$$d\widetilde{W}_t^j = q dW_t^i = \gamma \widetilde{W}_t^j dt - d\widetilde{U}_t^j + \phi_j d\mathcal{B}_t. \quad (\text{B.21})$$

Thus the scaled policy is feasible for firm j , including at the payout boundary, and its payoff is

$$\widetilde{p}_j(\mu_j w) = \mathbb{E} \left[\int_0^{\widetilde{\tau}_j} e^{-rs} (\mu_j ds - d\widetilde{U}_s^j) \right] = q \mathbb{E} \left[\int_0^{\tau_i} e^{-rs} (\mu_i ds - dU_s^i) \right] = qP_i(\mu_i w). \quad (\text{B.22})$$

Optimality therefore gives

$$P_j(\mu_j w) \geq qP_i(\mu_i w). \quad (\text{B.23})$$

Repeating the construction with i and j interchanged gives the reverse inequality. Hence

$$\frac{P_i(\mu_i w)}{\mu_i} = \frac{P_j(\mu_j w)}{\mu_j}, \quad w \geq 0. \quad (\text{B.24})$$

Equivalently, for every $x \geq 0$,

$$P_j(x) = q P_i(x/q). \quad (\text{B.25})$$

Differentiating for $x > 0$ yields $P'_j(x) = P'_i(x/q)$ and therefore

$$P'_i(\mu_i w) = P'_j(\mu_j w), \quad P'_i(0+) = P'_j(0+). \quad (\text{B.26})$$

Let

$$\mathcal{A}_\ell := \arg \max_{x \geq 0} P_\ell(x). \quad (\text{B.27})$$

The scaling identity implies $\mathcal{A}_j = q \mathcal{A}_i$. Consequently, if $W^{\ell,*}$ denotes the unique optimal initial promised utility, then

$$\frac{W^{i,*}}{\mu_i} = \frac{W^{j,*}}{\mu_j}. \quad (\text{B.28})$$

More generally, if maximizers are not unique, the normalized argmax sets coincide.

Finally, define the normalized processes, payout processes, and payout boundaries by

$$X_t^\ell := \frac{W_t^\ell}{\mu_\ell}, \quad A_t^\ell := \frac{U_t^\ell}{\mu_\ell}, \quad \bar{x}_\ell := \frac{\bar{W}_\ell}{\mu_\ell}. \quad (\text{B.29})$$

For $0 \leq t < \tau_\ell$, the optimal reflected payout policy satisfies

$$dX_t^\ell = \gamma X_t^\ell dt - dA_t^\ell + \frac{1}{\eta_\ell} d\mathcal{B}_t^\ell, \quad 0 \leq X_t^\ell \leq \bar{x}_\ell, \quad (\text{B.30})$$

where A^ℓ is nondecreasing and increases only when $X_t^\ell = \bar{x}_\ell$. The preceding scaling result gives

$$X_0^i = X_0^j, \quad \bar{x}_i = \bar{x}_j, \quad \eta_i = \eta_j. \quad (\text{B.31})$$

Thus the two normalized reflected diffusions have the same law. Since

$$\tau_\ell = \inf\{t \geq 0 : X_t^\ell = 0\}, \quad (\text{B.32})$$

τ_i and τ_j are equal in distribution. Under the common-Brownian coupling above, they are equal almost surely.

B.4 Comparative Statics of P

Centralized financing weakly reduces the volatility of the manager's promised value relative to decentralized financing.

Lemma 2. *The volatility of the conglomerate manager's promised value satisfies*

$$\phi_c := \sqrt{h(\beta_a^*, \beta_b^*)} \leq \phi_a + \phi_b. \quad (\text{B.33})$$

Proof of Lemma 2. This result follows from the inequality $h(\beta_a, \beta_b) \leq \sigma_a^2 \beta_a^2 + 2\sigma_a \sigma_b \beta_a \beta_b + \sigma_b^2 \beta_b^2 = (\sigma_a \beta_a + \sigma_b \beta_b)^2$ and the fact that $(\beta_a, \beta_b) = (\lambda_a, \lambda_b) \in \mathcal{C}$. Since (β_a^*, β_b^*) minimizes $h(\beta_a, \beta_b)$ over the IC set $\mathcal{C}$, we obtain

$$h(\beta_a^*, \beta_b^*) \leq h(\lambda_a, \lambda_b) \leq (\sigma_a \lambda_a + \sigma_b \lambda_b)^2 = (\phi_a + \phi_b)^2.$$

□

From Lemma 2, we obtain

$$\eta_c = \frac{\mu_c}{\phi_c} \geq \frac{\mu_a + \mu_b}{\phi_a + \phi_b} \geq \min \left\{ \frac{\mu_a}{\phi_a}, \frac{\mu_b}{\phi_b} \right\} = \min\{\eta_a, \eta_b\}. \quad (\text{B.34})$$

Equation (B.34) combines two distinct effects. Define $\eta_L := (\mu_a + \mu_b)/(\phi_a + \phi_b)$. The first inequality, $\eta_c \geq \eta_L$, captures the diversification gain from centralized contracting: the optimal centralized contract satisfies $\phi_c \leq \phi_a + \phi_b$ while leaving total expected cash flow $\mu_a + \mu_b$ unchanged. By contrast,

$$\eta_L = \frac{\phi_a}{\phi_a + \phi_b} \eta_a + \frac{\phi_b}{\phi_a + \phi_b} \eta_b \geq \min\{\eta_a, \eta_b\},$$

is an aggregation result: η_L is a ϕ -weighted average of the two project-level risk-adjusted productivities. Hence the conglomerate's risk-adjusted productivity is no lower than that of a project attaining $\min\{\eta_a, \eta_b\}$.

Below, we show that $P(W; \mu, \eta)$ is also monotone in both η and μ , a property that facilitates our characterization of the optimal organizational form.

Lemma 3. *For every $W \geq 0$, the investor's value function $P(W; \mu, \eta)$ is weakly increasing in both η and μ . For every $W > 0$, it is strictly increasing in each parameter, holding the other fixed. At $W = 0$, the boundary condition implies $P(0; \mu, \eta) = 0$ for all (μ, η) .*

Proof of Lemma 3. We first fix μ and compare two risk-adjusted productivities $\eta_1 > \eta_2$. Because $\eta = \mu/\phi$, this is equivalent to comparing volatility coefficients $\phi_1 < \phi_2$. By the comparison principle (Lemma 11), and using concavity so that $P'' \leq 0$, the smaller volatility coefficient weakly raises the value function. Thus

$$P(W; \mu, \eta_1) \geq P(W; \mu, \eta_2), \quad W \geq 0. \quad (\text{B.35})$$

Both value functions satisfy $P(0) = 0$, so equality holds at $W = 0$. Write $P_k(W) := P(W; \mu, \eta_k)$, let $\bar{W}_k$ denote its payout boundary, and set $D := P_1 - P_2$. Smooth fit, super contact, and the HJB equation give the linear payout tail

$$P_k(W) = \frac{\mu}{r} - \frac{\gamma - r}{r} \bar{W}_k - W, \quad W \geq \bar{W}_k.$$

Weak comparison on the common payout region, together with $\gamma > r$, therefore implies $\bar{W}_1 \leq \bar{W}_2$. On $(0, \bar{W}_1)$, strict concavity gives $P_2'' < 0$. Subtracting the two HJB equations gives

$$\frac{\phi_1^2}{2} D'' + \gamma W D' - r D = \frac{\phi_2^2 - \phi_1^2}{2} P_2'' < 0.$$

The strong maximum principle gives $D > 0$ on $(0, \bar{W}_1)$. Equality of the two payout boundaries is impossible: the Hopf boundary-point lemma would give $D'(\bar{W}_1) < 0$, whereas smooth fit gives $D'(\bar{W}_1) = 0$. Hence $\bar{W}_1 < \bar{W}_2$. On $[\bar{W}_1, \bar{W}_2)$, $D' = -1 - P_2' < 0$, while

$$D(\bar{W}_2) = \frac{\gamma - r}{r} (\bar{W}_2 - \bar{W}_1) > 0.$$

Beyond $\bar{W}_2$, the gap is constant because both value functions have slope -1 . Therefore,

$$P(W; \mu, \eta_1) > P(W; \mu, \eta_2), \quad W > 0. \quad (\text{B.36})$$

Next, fix η and let $\mu_1 > \mu_2 > 0$. Set $k := \mu_1/\mu_2 > 1$. By Proposition 2,

$$P(W; \mu_1, \eta) = k P(W/k; \mu_2, \eta). \quad (\text{B.37})$$

Concavity and $P(0; \mu_2, \eta) = 0$ imply

$$\begin{aligned} P(W/k; \mu_2, \eta) &\geq \frac{1}{k} P(W; \mu_2, \eta) + \left(1 - \frac{1}{k}\right) P(0; \mu_2, \eta) \\ &= \frac{1}{k} P(W; \mu_2, \eta). \end{aligned} \quad (\text{B.38})$$

Hence

$$P(W; \mu_1, \eta) \geq P(W; \mu_2, \eta), \quad W \geq 0. \quad (\text{B.39})$$

Equality necessarily holds at $W = 0$. For any $W > 0$, the value function is not affine on $[0, W]$ because it is strictly concave in the continuation region. The preceding concavity inequality is therefore strict, yielding

$$P(W; \mu_1, \eta) > P(W; \mu_2, \eta), \quad W > 0. \quad (\text{B.40})$$

Combining the two parts completes the proof. $\square$

Corollary 5. *Let $\ell \in \arg \min_{i \in \{a, b\}} \eta_i$, allowing for a tie. Then, for every $W \geq 0$,*

$$P(W; \mu_a + \mu_b, \eta_c) \geq P(W; \mu_\ell, \eta_\ell),$$

with equality at $W = 0$ and strict inequality for every $W > 0$.

Proof of Corollary 5. The proof follows from Lemma 3 and (B.34). For $W > 0$, because $\eta_c \geq \eta_\ell$ and $\mu_a + \mu_b > \mu_\ell$,

$$P(W; \mu_a + \mu_b, \eta_c) \geq P(W; \mu_a + \mu_b, \eta_\ell) > P(W; \mu_\ell, \eta_\ell).$$

At $W = 0$, both sides equal zero. $\square$

This corollary compares the conglomerate with a firm operating a minimum- η project alone; it does not by itself rank centralized financing against decentralized financing, whose value includes both stand-alone firms.

B.5 Proof of Proposition 3

Write $V_c(\eta_c) := \max_{W \geq 0} P(W; \mu_a + \mu_b, \eta_c)$ for the investor's value under centralized financing, and let $P_d^* := \max_{W \geq 0} P_a(W) + \max_{W \geq 0} P_b(W)$ be its decentralized counterpart. $P(W; \mu, \eta)$ is continuous in η for $W > 0$ and, by Lemma 3, strictly increasing in η . Hence V_c is continuous and strictly increasing, while P_d^* does not depend on η_c . The difference $V_c(\eta_c) - P_d^*$ therefore crosses zero at most once, which delivers uniqueness of the cutoff. For existence, $V_c(\eta_c) \rightarrow (\mu_a + \mu_b)/r$ as $\eta_c \rightarrow \infty$, the first-best value, whereas agency costs keep $P_d^* < (\mu_a + \mu_b)/r$; the difference is thus positive for η_c large enough.

It remains to show $\hat{\eta}_c \geq \eta_L$, equivalently $V_c(\eta_L) \leq P_d^*$. The value η_L is attained at $\rho = 1$, where $\phi_c = \phi_a + \phi_b$. Split the state in the ratio of the two agency measures: put

$x_i := \phi_i/\phi_c$, so that $x_a + x_b = 1$, and set $R_i(W) := P_i(x_i W)$. Because $R'_i(W) = x_i P'_i(x_i W)$ and $R''_i(W) = x_i^2 P''_i(x_i W)$, while $\phi_i^2/x_i^2 = \phi_c^2$, each R_i satisfies the stand-alone variational inequality with ϕ_c in place of ϕ_i . Their sum $Q := R_a + R_b$ therefore satisfies

$$\max \left\{ \frac{\phi_c^2}{2} Q''(W) + \gamma W Q'(W) + \mu_a + \mu_b - r Q(W), -Q'(W) - 1 \right\} \leq 0, \quad Q(0) = 0,$$

where the payout inequality uses $-Q' = x_a(-P'_a) + x_b(-P'_b) \leq x_a + x_b = 1$. So Q is a supersolution of the baseline conglomerate problem at $\rho = 1$, and the comparison argument behind Theorem 3 gives $Q \geq P_c$. Hence

$$V_c(\eta_L) = \max_W P_c(W) \leq \max_W Q(W) = \max_W (P_a(x_a W) + P_b(x_b W)) \leq P_d^*,$$

since each term is bounded by its own maximum.

B.6 Proof of Corollary 2

We easily observe that when $\rho = 1$,

$$\eta_c = \frac{\mu_a + \mu_b}{\sqrt{h(\beta_a^*, \beta_b^*)}} = \frac{\mu_a + \mu_b}{\phi_a + \phi_b} = \frac{\eta_a \phi_a + \eta_b \phi_b}{\phi_a + \phi_b} = \frac{\eta_a(\phi_a + \phi_b)}{\phi_a + \phi_b} = \eta_a = \eta_b. \quad (\text{B.41})$$

Since $\eta_a = \eta_b = \eta_c$, Proposition 2 (homogeneity) gives

$$\frac{\max_W P_c(W)}{\mu_c} = \frac{\max_W P_a(W)}{\mu_a} = \frac{\max_W P_b(W)}{\mu_b}. \quad (\text{B.42})$$

Therefore,

$$\begin{aligned} \max_W P_c(W) &= \frac{\mu_c}{\mu_a} \max_W P_a(W) = \frac{\mu_a + \mu_b}{\mu_a} \max_W P_a(W) \\ &= \max_W P_a(W) + \frac{\mu_b}{\mu_a} \max_W P_a(W) = \max_W P_a(W) + \max_W P_b(W). \end{aligned} \quad (\text{B.43})$$

It remains to establish strict preference for centralized financing at every admissible $\rho < 1$. Because (λ_a, λ_b) is feasible for the centralized contract and $\phi_a \phi_b > 0$,

$$\begin{aligned} \phi_c(\rho)^2 &= h(\beta_a^*(\rho), \beta_b^*(\rho)) \\ &\leq h(\lambda_a, \lambda_b) = \phi_a^2 + 2\rho \phi_a \phi_b + \phi_b^2 < (\phi_a + \phi_b)^2 = \phi_c(1)^2. \end{aligned}$$

Thus $\eta_c(\rho) > \eta_c(1) = \eta_a = \eta_b$. Holding $\mu_c = \mu_a + \mu_b$ fixed, the strict monotonicity in η established in Lemma 3 gives

$$P_c(W; \rho) > P_c(W; 1), \quad W > 0.$$

Let $W_1^* > 0$ maximize $P_c(\cdot; 1)$; positivity follows from $P_c(0; 1) = 0$ and (B.42). Consequently,

$$\begin{aligned} \max_W P_c(W; \rho) &\geq P_c(W_1^*; \rho) > P_c(W_1^*; 1) \\ &= \max_W P_c(W; 1) = \max_W P_a(W) + \max_W P_b(W), \quad \rho < 1. \end{aligned}$$

Hence, the cutoff correlation is $\hat{\rho} = 1$ in this case.

B.7 Proof of Corollary 3

We prove Corollary 3 along the ceteris-paribus path stated in the corollary, holding all primitives listed there fixed and varying only $\sigma_b > 0$. Thus $\eta_a = \mu_a/(\lambda_a \sigma_a)$ is fixed, whereas $\eta_b = \mu_b/(\lambda_b \sigma_b)$ tends to $+\infty$ as $\sigma_b \downarrow 0$ and to 0 as $\sigma_b \rightarrow \infty$. It is enough to establish decentralization at $\rho = -1$, the correlation most favorable to centralization; decentralization there implies decentralization at every admissible correlation.

We first examine the endpoint $\sigma_b = 0$ at $\rho = -1$, understood as the limit $\sigma_b \downarrow 0$. At this endpoint,

$$P_b(W) = \frac{\mu_b}{r} - W, \quad \max_W P_b(W) = \frac{\mu_b}{r}. \quad (\text{B.44})$$

In this case, P_c satisfies

$$\max \left\{ \frac{\phi_a^2}{2} P_c'' + \gamma W P_c' + \mu_a + \mu_b - r P_c, -P_c' - 1 \right\} = 0, \quad P_c(0) = 0, \quad (\text{B.45})$$

while P_a satisfies

$$\max \left\{ \frac{\phi_a^2}{2} P_a'' + \gamma W P_a' + \mu_a - r P_a, -P_a' - 1 \right\} = 0, \quad P_a(0) = 0. \quad (\text{B.46})$$

Define

$$\tilde{P}(W) := P_a(W) + \frac{\mu_b}{r}. \quad (\text{B.47})$$

Then

$$\max \left\{ \frac{\phi_a^2}{2} \tilde{P}'' + \gamma W \tilde{P}' + \mu_a + \mu_b - r \tilde{P}, -\tilde{P}' - 1 \right\} = 0, \quad \tilde{P}(0) = \frac{\mu_b}{r} > 0. \quad (\text{B.48})$$

Thus, $\tilde{P}$ is a supersolution to (B.45). Set $D := \tilde{P} - P_c$, and let $\bar{W}_a$ and $\bar{W}_c$ denote the payout boundaries of $\tilde{P}$ and P_c . Weak comparison gives $D \geq 0$. On the common payout tail, $D = (\gamma - r)(\bar{W}_c - \bar{W}_a)/r$, so $\bar{W}_a \leq \bar{W}_c$. On $(0, \bar{W}_a)$, D solves the common homogeneous continuation ODE and $D(0) = \mu_b/r > 0$; the strong maximum principle gives $D > 0$. Equality is impossible: the Hopf boundary-point lemma gives $D'(\bar{W}_a) < 0$, contradicting the smooth-fit value $D'(\bar{W}_a) = 0$. Hence $\bar{W}_a < \bar{W}_c$. On $[\bar{W}_a, \bar{W}_c)$, $D' = -1 - P'_c < 0$, while $D(\bar{W}_c) = (\gamma - r)(\bar{W}_c - \bar{W}_a)/r > 0$; beyond $\bar{W}_c$, this positive gap is constant. Together with $D(0) > 0$, this proves

$$\tilde{P}(W) > P_c(W), \quad W \geq 0. \quad (\text{B.49})$$

Let W_c^* maximize P_c . Evaluating the strict comparison there gives

$$\begin{aligned} \max_W P_c(W) &= P_c(W_c^*) < \tilde{P}(W_c^*) \leq \max_W \tilde{P}(W) \\ &= \max_W P_a(W) + \max_W P_b(W). \end{aligned} \quad (\text{B.50})$$

At the other end of the same fixed-primitives path, we next consider small η_b (large σ_b).

Lemma 4. *We have the following limits:*

$$\lim_{\sigma_b \rightarrow \infty} \max_W P_b(W; \sigma_b) = 0, \quad \lim_{\sigma_b \rightarrow \infty} \max_W P_c(W; \sigma_b) = 0. \quad (\text{B.51})$$

As a result, for all sufficiently large σ_b ,

$$\max_W P_c(W) < \max_W P_a(W) + \max_W P_b(W). \quad (\text{B.52})$$

Both endpoint arguments deliver strict inequalities in the limit. Since $\max_W P_b$ and $\max_W P_c$ are continuous in σ_b ($P(W; \mu, \eta)$ is continuous in η , and ϕ_c is continuous in σ_b through (35)), the strict inequality persists for all sufficiently small and all sufficiently large σ_b , which yields the constant K in Corollary 3.

Proof of Lemma 4. Fix $\mu > 0$ and write $P_\phi(W) := P(W; \mu, \mu/\phi)$ for the reduced value function with volatility coefficient $\phi > 0$. Let $\bar{W}_\phi$ denote its payout boundary. Immediate payment of W followed by liquidation gives the lower bound below; nonnegative compensation and the present value of cash flows give the upper bound:

$$-W \leq P_\phi(W) \leq \frac{\mu}{r}, \quad W \geq 0. \quad (\text{B.53})$$

Smooth fit, super contact, and the continuation equation at $\bar{W}_\phi$ imply

$$P_\phi(\bar{W}_\phi) = \frac{\mu - \gamma \bar{W}_\phi}{r}, \quad 0 < \bar{W}_\phi \leq \frac{\mu}{\gamma - r}, \quad (\text{B.54})$$

where the boundary bound follows from $P_\phi(\bar{W}_\phi) \geq -\bar{W}_\phi$.

Concavity and $P_\phi(0) = 0$ imply $WP'_\phi(W) \leq P_\phi(W)$. Hence, throughout the continuation region $0 < W < \bar{W}_\phi$, the ODE gives

$$\begin{aligned} 0 \leq -P''_\phi(W) &= \frac{2}{\phi^2} (\mu + \gamma W P'_\phi(W) - r P_\phi(W)) \\ &\leq \frac{2}{\phi^2} (\mu + (\gamma - r) P_\phi(W)) \leq \frac{2\gamma\mu}{r\phi^2}. \end{aligned} \quad (\text{B.55})$$

Integrating backward from $P'_\phi(\bar{W}_\phi) = -1$ yields

$$0 \leq P'_\phi(W) + 1 = \int_W^{\bar{W}_\phi} [-P''_\phi(s)] ds \leq \frac{2\gamma\mu}{r\phi^2} (\bar{W}_\phi - W), \quad 0 < W < \bar{W}_\phi. \quad (\text{B.56})$$

Since $P_\phi(0) = 0$ and $P_\phi(W) + W$ is constant for $W \geq \bar{W}_\phi$, a second integration gives the uniform estimate

$$0 \leq P_\phi(W) + W \leq \frac{\gamma\mu\bar{W}_\phi^2}{r\phi^2} \leq \frac{\gamma\mu^3}{r(\gamma - r)^2\phi^2}, \quad W \geq 0. \quad (\text{B.57})$$

Thus $P_\phi(W) \rightarrow -W$ uniformly on $[0, \infty)$. Moreover, $P_\phi(0) = 0$ implies

$$0 \leq \max_{W \geq 0} P_\phi(W) \leq \frac{\gamma\mu^3}{r(\gamma - r)^2\phi^2} \rightarrow 0 \quad \text{as } \phi \rightarrow \infty. \quad (\text{B.58})$$

For firm b , apply this estimate with $\mu = \mu_b$ and $\phi = \phi_b = \lambda_b \sigma_b$. Since $\lambda_b > 0$ is fixed, $\phi_b \rightarrow \infty$ as $\sigma_b \rightarrow \infty$. This proves the first limit in (B.51) and also establishes

$$P_\infty(W) := \lim_{\sigma_b \rightarrow \infty} P_b(W; \sigma_b) = -W, \quad W \geq 0. \quad (\text{B.59})$$

For the conglomerate, $\mu_c = \mu_a + \mu_b$ is fixed. Once $\phi_b > \sigma_a$, the IC constraints give $\beta_b \geq \lambda_b$ and $\beta_a \leq \beta_b/\lambda_b$, so

$$\sigma_b \beta_b - \sigma_a \beta_a \geq \beta_b \left(\sigma_b - \frac{\sigma_a}{\lambda_b} \right) \geq \phi_b - \sigma_a > 0.$$

Consequently, for every feasible sensitivity pair and every $\rho \in [-1, 1]$,

$$\begin{aligned} h(\beta_a, \beta_b) &= (\sigma_b \beta_b - \sigma_a \beta_a)^2 + 2(1 + \rho) \sigma_a \sigma_b \beta_a \beta_b \\ &\geq (\phi_b - \sigma_a)^2. \end{aligned} \tag{B.60}$$

Minimizing over the IC set gives $\phi_c \geq \phi_b - \sigma_a \rightarrow \infty$. Applying (B.58) with $(\mu, \phi) = (\mu_c, \phi_c)$ proves the second limit in (B.51). $\square$

C Proofs for Section 5

C.1 Proof of Proposition 4

We show that centralized financing weakly dominates decentralized financing.

Since the conglomerate volatility $h(\beta_a^*, \beta_b^*)$ is increasing in ρ and F is concave by Theorem 2, without loss of generality we restrict attention to the worst-case correlation $\rho = 1$.

When $\rho = 1$, the two Brownian motions coincide:

$$\mathcal{Z}_t^a = \mathcal{Z}_t^b = \mathcal{Z}_t, \tag{C.1}$$

and therefore generate the same filtration. In this case, the optimal payout strategies for the two stand-alone firms can be replicated within the conglomerate. Intuitively, because both projects are driven by the same shock, the investor can infer each stand-alone promised utility W_t^a and W_t^b from the conglomerate's promised utility W_t^c .

Formally, set $\rho = 1$ and let $(W^{a,*}, W^{b,*})$ denote the optimal initial promised utilities in the two stand-alone problems. Their evolutions before payout or liquidation satisfy

$$dW_t^a = \gamma W_t^a dt + \lambda_a \sigma_a d\mathcal{Z}_t, \quad W_0^a = W^{a,*}, \tag{C.2}$$

$$dW_t^b = \gamma W_t^b dt + \lambda_b \sigma_b d\mathcal{Z}_t, \quad W_0^b = W^{b,*}. \tag{C.3}$$

Consider a conglomerate firm that promises

$$W_0^c = W^{a,*} + W^{b,*}. \tag{C.4}$$

Before payout or spinoff, choose $(\beta_a^*, \beta_b^*) = (\lambda_a, \lambda_b)$, and its promised-utility dynamics are

$$dW_t^c = \gamma W_t^c dt + (\lambda_a \sigma_a + \lambda_b \sigma_b) d\mathcal{Z}_t. \tag{C.5}$$

Since the diffusion coefficient is nonzero, observing dW_t^c reveals $d\mathcal{Z}_t$ pathwise. Consequently,

the investor can reconstruct the evolutions in (C.2)–(C.3) directly from W_t^c .

Let dU_t^a and dU_t^b denote the stand-alone optimal payout strategies (triggered when W_t^a and W_t^b reach $\bar{W}^a$ and $\bar{W}^b$), and let τ_a and τ_b be the respective liquidation times (first hitting times of 0). Because W_t^a and W_t^b are measurable with respect to the filtration generated by W_t^c , the stand-alone optimal strategies are admissible for the conglomerate.

Define the combined strategy

$$U_t^c = U_t^a + U_t^b, \quad \tau_c = \max\{\tau_a, \tau_b\}. \quad (\text{C.6})$$

This keeps $W_t^c = W_t^a + W_t^b \geq 0$ until both projects are liquidated and is therefore feasible. At $\tau_a \wedge \tau_b$, the division whose stand-alone promised value has reached zero is spun off; its stand-alone value at that point is $P_i(0) = 0$, so this is equivalent to liquidating it. The other division continues under its own stand-alone contract until its liquidation time $\tau_a \vee \tau_b$. Moreover, the total expected payoff equals

$$P_a(W^{a,*}) + P_b(W^{b,*}) = \max_W P_a(W) + \max_W P_b(W). \quad (\text{C.7})$$

Thus when $\kappa = 0$, centralized financing can replicate the decentralized value and is therefore optimal.

C.2 Proof of Proposition 5

The following corollary records an unconditional fact: whenever separation occurs, it is always the lower- η division that is spun off, never the higher- η division—and this holds regardless of whether the hypothesis $\eta_c < \max\{\eta_a, \eta_b\}$ of Proposition 5 holds.

Corollary 6. *If $\eta_a \neq \eta_b$, let H denote the higher- η division and ℓ the lower- η division. Then $F(W) > P_\ell(W)$ for every $W > 0$. If $\eta_a = \eta_b$, then $F(W) > \max\{P_a(W), P_b(W)\}$ for every $W > 0$. Consequently, keeping only a lower- η division—equivalently, spinning off a higher- η division—is never optimal.*

Proof of Corollary 6. The conglomerate's flow productivity satisfies $\mu_c = \mu_a + \mu_b > \mu_a, \mu_b$, and $\eta_c \geq \min\{\eta_a, \eta_b\}$ (see (B.34)). Under the maintained asymmetric labeling $\eta_b > \eta_a$, we have $\mu_c > \mu_a$ and $\eta_c \geq \eta_a$, so Lemma 3 (strict in μ) gives

$$F(W) \geq P_c(W) = P(W; \mu_c, \eta_c) > P(W; \mu_a, \eta_a) = P_a(W), \quad W > 0, \quad (\text{C.8})$$

which proves Corollary 6: the conglomerate is never spun off into keeping the lower- η division a alone. $\square$

Now suppose in addition $\eta_c \geq \eta_b = \max\{\eta_a, \eta_b\}$, i.e., the hypothesis of the first part of Proposition 5 fails. Since $\mu_c > \mu_b$ as well, the same argument with b in place of a gives

$$F(W) \geq P_c(W) > P_b(W), \quad W > 0. \quad (\text{C.9})$$

At $W = 0$, all three value functions equal zero. Combined with the bound on P_a above, F strictly exceeds both P_a and P_b for every $W > 0$: neither obstacle ever binds, so the conglomerate is never spun off and $\tau_a = \tau_b$. This covers both $\eta_a \neq \eta_b$ with $\eta_c \geq \max\{\eta_a, \eta_b\}$ and $\eta_a = \eta_b$ (for which $\eta_c \geq \eta_L = \eta_a = \eta_b$ automatically, by (B.34)). $\square$

Suppose $\eta_a \neq \eta_b$ and $\eta_c < \max\{\eta_a, \eta_b\}$; maintain $\eta_b > \eta_a$ as above, so $\eta_a < \eta_b$ strictly and $\eta_c < \eta_b$. Proposition 6 below—whose proof uses only Corollary 6 above, not Proposition 5 itself—establishes a unique cutoff $\widehat{W} \in (0, \overline{W})$ such that the lower- η division a is spun off the first time W_t falls to $\widehat{W}$: $\tau_a = \inf\{t \geq 0 : W_t \leq \widehat{W}\}$. Since $\widehat{W} > 0$, this occurs at a finite time. After the spinoff, the surviving division b continues alone from promised utility $\widehat{W} > 0$, and the conglomerate's eventual liquidation, $\tau_a \vee \tau_b$, occurs only once the survivor's own promised utility subsequently reaches 0, strictly after τ_a . Hence $\tau_a \wedge \tau_b < \tau_a \vee \tau_b$ almost surely, with the lower- η division spun off first.

C.3 Proof of Proposition 6

Suppose $\eta_b > \eta_a$; write $\ell := a$ for the lower- η division that is spun off and $H := b$ for the surviving, higher- η division, and let P_c denote the value of operating the conglomerate without exercising the spinoff option. Under $\kappa = 0$, $\mu_c = \mu_H + \mu_\ell$, and never exercising the option is feasible, so

$$F(W) \geq P_c(W), \quad W \geq 0. \quad (\text{C.10})$$

By Corollary 6, P_H is the only possible positive spinoff obstacle. The hypothesis $\eta_c < \max\{\eta_a, \eta_b\}$ of Proposition 6 gives $\eta_c < \eta_H$ directly, so it remains only to prove sufficiency: existence and uniqueness of the cutoff.

(i). We first prove that the spinoff region for a is nonempty. We begin by recording, in a lemma, how the slope of the value function at zero depends on η .

Lemma 5 (Boundary slope and risk-adjusted productivity). *There exists a strictly increasing function $s : (0, \infty) \rightarrow \mathbb{R}$ such that $P'(0+; \mu, \eta) = s(\eta)$ for every $\mu > 0$ and $\eta > 0$.*

Proof of Lemma 5. Proposition 2 gives $P(W; \mu, \eta) = \mu P(W/\mu; 1, \eta)$. Taking the right derivative at zero yields $P'(0+; \mu, \eta) = P'(0+; 1, \eta)$, which defines $s(\eta)$ and proves independence from μ . To prove strict monotonicity, fix $\mu > 0$ and take $\eta_1 > \eta_2 > 0$. Lemma 3 gives

$P(W; \mu, \eta_1) > P(W; \mu, \eta_2)$ for $W > 0$, while both functions vanish at zero, so $s(\eta_1) \geq s(\eta_2)$. If equality held, the one-project ODE at $W = 0$, with $\phi_i = \mu/\eta_i$, would give

$$P''(0+; \mu, \eta_i) = -\frac{2\mu}{\phi_i^2} = -\frac{2\eta_i^2}{\mu}, \quad (\text{C.11})$$

and a second-order expansion would imply $P(W; \mu, \eta_1) < P(W; \mu, \eta_2)$ for all sufficiently small $W > 0$, a contradiction. Hence $s(\eta_1) > s(\eta_2)$. $\square$

By Lemma 5 and $\eta_H > \eta_c$, we have

$$P'_H(0+) > P'_c(0+), \quad (\text{C.12})$$

and therefore $P_H(W) > P_c(W)$ for all sufficiently small $W > 0$. If the spinoff contact set were empty on $(0, \infty)$, the obstacle would never bind and uniqueness of the no-spinoff problem in Lemma 9 and Theorem 1 would give $F = P_c$. This would contradict both $F \geq P_H$ and the preceding strict inequality. Thus a positive contact point exists.

(ii) Next, we prove that the contact set has a unique boundary. Below the payout boundary, define the conglomerate continuation operator and the residual from delaying spinoff by

$$\mathcal{A}_c f(W) := \frac{\phi_c^2}{2} f''(W) + \gamma W f'(W) - r f(W) + \mu_c, \quad (\text{C.13})$$

$$\Psi(W) := \mathcal{A}_c P_H(W). \quad (\text{C.14})$$

The following lemma summarizes the single-crossing property of Ψ and the resulting restriction on contact, under conditions that do not require $\kappa = 0$.

Lemma 6 (Single crossing and contact exclusion). *Suppose $\eta_H > \eta_c$ and $\mu_H < \mu_c$.*

- (i) *There exists a unique point $W_0 \in (0, \overline{W}_H)$ such that $\Psi(W_0) = 0$. Moreover, $\Psi(W) < 0$ for $0 < W < W_0$, and $\Psi(W) > 0$ for $W_0 < W \leq \overline{W}_H$. Thus Ψ crosses zero exactly once, from below, at W_0 .*
- (ii) *If P_H is an available spinoff payoff for the conglomerate value function F , then $F(W) > P_H(W)$ for every $W > W_0$. Consequently, every contact point between F and P_H lies in $[0, W_0]$.*

Proof of Lemma 6. (i) On the stand-alone continuation region $[0, \overline{W}_H]$, the ODE for P_H yields

$$\Psi(W) = \mu_c - \mu_H - \frac{\phi_c^2 - \phi_H^2}{2} m(W), \quad m(W) := -P_H''(W). \quad (\text{C.15})$$

Here $m(W) > 0$ for $0 < W < \bar{W}_H$, and $m(\bar{W}_H) = 0$. Because $\eta_c < \eta_H$ and $\mu_H < \mu_c$,

$$\frac{\phi_c}{\phi_H} = \frac{\mu_c}{\mu_H} \frac{\eta_H}{\eta_c} > \frac{\mu_c}{\mu_H} > 1. \quad (\text{C.16})$$

Using $P_H''(0+) = -2\mu_H/\phi_H^2$ and the super-contact condition $P_H''(\bar{W}_H) = 0$, we obtain

$$\Psi(0+) = \mu_c - \frac{\phi_c^2}{\phi_H^2} \mu_H < 0, \quad \Psi(\bar{W}_H) = \mu_c - \mu_H > 0. \quad (\text{C.17})$$

Differentiating the stand-alone ODE twice shows that m satisfies

$$\frac{\phi_H^2}{2} m''(W) + \gamma W m'(W) + (2\gamma - r)m(W) = 0. \quad (\text{C.18})$$

At any interior stationary point of m , that is $m'(W) = 0$ with $0 < W < \bar{W}_H$,

$$m''(W) = -\frac{2(2\gamma - r)}{\phi_H^2} m(W) < 0, \quad (\text{C.19})$$

where the inequality follows from $m > 0$ and $\gamma > r$. Hence m' has at most one zero. The once-differentiated ODE gives $m'(0+) = 2(\gamma - r)P_H'(0+)/\phi_H^2$ and $m'(\bar{W}_H-) = -2(\gamma - r)/\phi_H^2 < 0$; the first sign is unrestricted. Hence m is either strictly decreasing or first increasing and then decreasing (if $m'(0+) = 0$, the equation for m gives $m''(0+) < 0$). Moreover, the endpoint signs above are equivalent to $m(0+) > q := 2(\mu_c - \mu_H)/(\phi_c^2 - \phi_H^2) > m(\bar{W}_H) = 0$. Either shape crosses q exactly once, from above. Since $\Psi(W) = \mu_c - \mu_H - (\phi_c^2 - \phi_H^2)m(W)/2$ and $\phi_c^2 - \phi_H^2 > 0$, Ψ crosses zero exactly once, from below. The endpoint signs place this unique zero W_0 in $(0, \bar{W}_H)$ and give the stated signs on either side.

(ii) For $W \in (W_0, \bar{W}_H)$, operating the conglomerate for a short interval h and then spinning off, with liquidation if zero is reached first, yields $P_H(W) + \Psi(W)h + o(h) > P_H(W)$ for sufficiently small $h > 0$. Since this policy is feasible, $F(W) > P_H(W)$.

At $W = \bar{W}_H$, operate the conglomerate for a short interval, using payouts to reflect promised utility at $\bar{W}_H$, and then spin off, again liquidating if zero is reached first. The payout term in Itô's formula for the total payoff is $-[1 + P_H'(\bar{W}_H)]dU_t = 0$ by smooth fit. The net gain over immediate spinoff is therefore $\Psi(\bar{W}_H)h + o(h) = (\mu_c - \mu_H)h + o(h) > 0$ for sufficiently small $h > 0$, so $F(\bar{W}_H) > P_H(\bar{W}_H)$.

Finally, for $W > \bar{W}_H$, an immediate payout of $W - \bar{W}_H$ followed by an optimal policy from $\bar{W}_H$ is feasible. Since P_H is linear with slope -1 above $\bar{W}_H$,

$$\begin{aligned} F(W) &\geq F(\bar{W}_H) - (W - \bar{W}_H) \\ &> P_H(\bar{W}_H) - (W - \bar{W}_H) = P_H(W). \end{aligned}$$

Combining the three cases proves that contact is impossible for every $W > W_0$. $\square$

We now use Lemma 6(ii) to characterize the contact set. Let $D := F - P_H \geq 0$. By the lemma, every contact point lies in $[0, W_0]$.

The contact set is closed by continuity and contains a positive point, so let $\widehat{W}$ denote its largest element. We will prove by contradiction that there can be no gap between zero and the first positive contact point, or between any two contact points. Otherwise, on such a continuation component $(a, b) \subset (0, W_0)$, standard one-dimensional obstacle regularity gives smooth fit at the endpoints, the payout constraint is inactive, and D satisfies

$$\frac{\phi_c^2}{2} D''(W) + \gamma W D'(W) - r D(W) = -\Psi(W) \geq 0, \quad (\text{C.20})$$

with $D(a) = D(b) = 0$ and $D > 0$ inside. At an interior positive maximum, however, the left-hand side is strictly negative, contradicting the maximum principle. Hence the spinoff region is the single lower interval

$$\mathbf{S}_\ell = (0, \widehat{W}]. \quad (\text{C.21})$$

Because $\phi_c > 0$, the boundary $\widehat{W}$ is regular, and value matching and smooth pasting give

$$F(\widehat{W}) = P_H(\widehat{W}), \quad F'(\widehat{W}) = P'_H(\widehat{W}). \quad (\text{C.22})$$

Since $\widehat{W} \leq W_0 < \overline{W}_H$, strict concavity of P_H implies $P'_H(\widehat{W}) > -1$. Concavity of F and Theorem 2 then imply $\widehat{W} < \overline{W}$. Finally, continuity of the promised-utility process gives

$$\tau_\ell = \inf\{t \geq 0 : W_t \leq \widehat{W}\}. \quad (\text{C.23})$$

This proves the result.

D Proofs for Section 6

D.1 Proof of Proposition 7

Proposition 7 in Section 6.1 states the five-region solution. We first record the single-crossing property of the two stand-alone values on which it rests; the proof of the proposition follows. The proof does not use the assumption that both divisions are spun off, and Appendix D.2 records the configurations with fewer than five regions.

Lemma 7 (Single crossing of the stand-alone values). *Suppose $\eta_b > \eta_a$ and $\mu_b < \mu_a$. Then $P_b - P_a$ changes sign at most once on $(0, \infty)$: there is at most one $W^* > 0$ with $P_b(W^*) =$*

$P_a(W^*)$ at which $P_b - P_a$ changes sign, and in that case $P_b > P_a$ on $(0, W^*)$ and $P_a > P_b$ on (W^*, ∞) . Such a crossing exists if and only if $P_a(\overline{W}_a) + \overline{W}_a > P_b(\overline{W}_b) + \overline{W}_b$, that is, if and only if division a's stand-alone enterprise value is the higher one at high promised values.

Proof of Lemma 7. (i) **We prove that the stand-alone payout boundaries satisfy $\overline{W}_b < \overline{W}_a$.** For a one-project problem, set

$$x := \frac{W}{\phi}, \quad p(x; \eta) := \frac{P(\phi x; \mu, \eta)}{\phi}, \quad b(\eta) := \frac{\overline{W}}{\phi}. \quad (\text{D.1})$$

On $[0, b(\eta)]$, the normalized value solves

$$\frac{1}{2}p''(x) + \gamma x p'(x) - r p(x) + \eta = 0, \quad p(0) = 0, \quad p'(b) = -1, \quad p''(b) = 0. \quad (\text{D.2})$$

Let u solve

$$\frac{1}{2}u''(x) + \gamma x u'(x) + (\gamma - r)u(x) = 0, \quad u(0) = 1, \quad u'(0) = 0. \quad (\text{D.3})$$

If v is the solution with $v(0) = 0$ and $v'(0) = 1$, their Wronskian is $uv' - u'v = e^{-\gamma x^2}$. Applying the two conditions at b to $p' = Au + Bv$, and using $p''(0) = -2\eta$, gives

$$\eta(b) = -\frac{1}{2}e^{\gamma b^2}u'(b). \quad (\text{D.4})$$

To determine the sign of this expression, let $\alpha := (\gamma - r)/(2\gamma) \in (0, 1/2)$. The Euler representation

$$u(x) = K_\alpha \int_0^1 e^{-\gamma x^2 t} t^{\alpha-1} (1-t)^{-1/2-\alpha} dt, \quad K_\alpha > 0, \quad (\text{D.5})$$

implies $u(x) > 0$ and $u'(x) < 0$ for $x > 0$. Differentiating (D.4) and using the ODE for u yields

$$\begin{aligned} \eta'(b) &= (\gamma - r)e^{\gamma b^2}u(b) > 0, \\ b\eta'(b) - \eta(b) &= \frac{1}{2}e^{\gamma b^2}[u'(b) + 2(\gamma - r)bu(b)] > 0. \end{aligned} \quad (\text{D.6})$$

For the last inequality, define $H(b) := u'(b) + 2(\gamma - r)bu(b)$. Then

$$H(0) = 0, \quad H'(b) = -2rbu'(b) > 0.$$

It follows from (D.6) that $b(\eta)/\eta$ is strictly decreasing in η . Since

$$\overline{W}_i = \phi_i b(\eta_i) = \mu_i \frac{b(\eta_i)}{\eta_i}, \quad (\text{D.7})$$

the conflicting rankings $\eta_b > \eta_a$ and $\mu_b < \mu_a$ imply

$$\overline{W}_b < \overline{W}_a. \quad (\text{D.8})$$

This ordering identifies $(0, \overline{W}_b)$ as the common continuation domain on which we compare the two curvature functions.

(ii) Defining

$$m_i(W) := -P_i''(W) > 0, \quad 0 < W < \overline{W}_i, \quad i \in \{a, b\}, \quad (\text{D.9})$$

we prove that the stand-alone curvature ratio $m_b/m_a \geq 0$ crosses the level one exactly once, from above to below. The rankings over η and μ imply $\phi_b = \mu_b/\eta_b < \mu_a/\eta_a = \phi_a$. Twice differentiating the one-project ODE gives

$$\frac{\phi_i^2}{2} m_i''(W) + \gamma W m_i'(W) + (2\gamma - r) m_i(W) = 0. \quad (\text{D.10})$$

The boundary conditions imply

$$m_i(0+) = \frac{2\mu_i}{\phi_i^2}, \quad m_i(\overline{W}_i) = 0, \quad m_i'(\overline{W}_i-) = -\frac{2(\gamma - r)}{\phi_i^2}. \quad (\text{D.11})$$

Put $z_i := m_i'/m_i$. Its Riccati equation is

$$z_i' = -z_i^2 - \frac{2\gamma W}{\phi_i^2} z_i - \frac{2(2\gamma - r)}{\phi_i^2}. \quad (\text{D.12})$$

First, we prove that each z_i is strictly decreasing. Indeed, from the differential equation above, we see $z_i'(0) < 0$ and

$$z_i(W) \sim -(\overline{W}_i - W)^{-1} \text{ near } \overline{W}_i-. \quad (\text{D.13})$$

If a connected component (x, y) of $\{z_i' > 0\}$ existed, then differentiating (D.12) would give

$$z_i'' = -\frac{2\gamma}{\phi_i^2} z_i \quad \text{whenever } z_i' = 0. \quad (\text{D.14})$$

At the left endpoint, $z_i''(x) \geq 0$ implies $z_i(x) < 0$; at the right endpoint, $z_i''(y) \leq 0$ implies

$z_i(y) > 0$. Since $z'_i > 0$ on (x, y) , z_i would cross zero upward there; but (D.12) gives $z'_i < 0$ whenever $z_i = 0$, a contradiction.

Second, define $q := z_b - z_a$ on $(0, \bar{W}_b)$. We prove that q is either nonpositive throughout or changes sign at most once, from positive to negative.

We have from (i) and (D.13) that

$$\lim_{W \uparrow \bar{W}_b} q(W) = -\infty. \quad (\text{D.15})$$

At any zero of q , with common value $z = z_a = z_b$, subtracting the two Riccati equations (D.12) gives

$$q' = -2 \left(\frac{1}{\phi_b^2} - \frac{1}{\phi_a^2} \right) [\gamma W z + (2\gamma - r)]. \quad (\text{D.16})$$

Let $T(W) := -(2\gamma - r)/(\gamma W)$, which is strictly increasing. If q crossed from negative to positive at some x , then (D.16) would imply $z_b(x) \leq T(x)$. At any later zero $y > x$, strict decrease of z_b would give $z_b(y) < z_b(x) \leq T(x) < T(y)$ and hence $q'(y) > 0$. Thus q could never subsequently cross from positive to negative, contradicting its limit at $\bar{W}_b$. Therefore q has no negative-to-positive crossing. Since its terminal limit is negative, the claim follows.

Third, since $(\frac{m_b}{m_a})' = \frac{m_b}{m_a} q$, it follows that m_b/m_a is either nonincreasing throughout or first increasing and then decreasing. Moreover, from (D.11) and (D.13),

$$\frac{m_b(0+)}{m_a(0+)} = \frac{\eta_b \phi_a}{\eta_a \phi_b} > 1, \quad \lim_{W \uparrow \bar{W}_b} \frac{m_b(W)}{m_a(W)} = 0. \quad (\text{D.17})$$

Hence m_b/m_a crosses the level one exactly once.

(iii) We deduce that $P_b - P_a$ has at most one positive sign-changing zero W^* . By Lemma 5, $P'_i(0+) = s(\eta_i)$, where s is strictly increasing. Put $D := P_b - P_a$. On $(0, \bar{W}_b)$, Step (ii) implies that $D'' = m_a - m_b$ changes sign exactly once, from negative to positive. On $[\bar{W}_b, \bar{W}_a)$, $P''_b = 0$ and $P''_a = -m_a < 0$, so $D'' = m_a > 0$. Thus D'' has exactly one sign change on the full domain, and D' first decreases and then increases. Moreover,

$$D'(0+) = s(\eta_b) - s(\eta_a) > 0, \quad D'(\bar{W}_b) = -1 - P'_a(\bar{W}_b) < 0, \quad D'(\bar{W}_a) = 0. \quad (\text{D.18})$$

Thus D' crosses zero exactly once on its decreasing branch and remains negative on its increasing branch until it reaches zero at $\bar{W}_a$. Since $D(0) = 0$, D first rises, then falls, and is constant after $\bar{W}_a$. Therefore $P_b - P_a$ has at most one positive sign-changing zero. Writing

the stand-alone enterprise-value plateau as

$$C_i := P_i(\overline{W}_i) + \overline{W}_i = \frac{\mu_i - (\gamma - r)\overline{W}_i}{r}, \quad (\text{D.19})$$

we obtain a unique transverse crossing W^* precisely when $C_a > C_b$. If $C_a < C_b$, the values never cross; if $C_a = C_b$, their linear payout tails coincide without a sign reversal. $\square$

We then return to the proof of Proposition 7.

(i) We prove that contact with each obstacle is either absent or confined below a unique residual cutoff. Contact with P_a corresponds to spinning off division b , whereas contact with P_b corresponds to spinning off division a . Define

$$\mathcal{A}_c f(W) := \frac{\phi_c^2}{2} f''(W) + \gamma W f'(W) - r f(W) + \mu_c, \quad \Psi_i := \mathcal{A}_c P_i. \quad (\text{D.20})$$

Let P_c be the value when the conglomerate is operated without the spinoff option. If $\eta_c \geq \eta_i$, then $\mu_c > \mu_i$ and Lemma 3 gives

$$F(W) \geq P_c(W) > P_i(W), \quad W > 0. \quad (\text{D.21})$$

Thus obstacle P_i never binds. It remains to consider $\eta_c < \eta_i$. Since $\mu_c > \mu_i$, Lemma 6(i) applies with $H = i$. Hence Ψ_i has a unique zero $\zeta_i \in (0, \overline{W}_i)$, with $\Psi_i(W) < 0$ for $0 < W < \zeta_i$ and $\Psi_i(W) > 0$ for $\zeta_i < W \leq \overline{W}_i$. Because P_i is an available spinoff payoff, Lemma 6(ii) also gives

$$\{W > 0 : F(W) = P_i(W)\} \subset (0, \zeta_i]. \quad (\text{D.22})$$

Standard one-dimensional smooth fit and $\zeta_i < \overline{W}_i$ imply $F' = P'_i > -1$ at every spinoff boundary, so all spinoff contact lies strictly below the conglomerate payout boundary $\overline{W}$.

(ii) We prove that each spinoff contact set is connected. Consider two contact points with the same obstacle P_i , separated by a component (x, y) on which $F > P_i$. At a positive contact point, P_i must be the upper obstacle. Lemma 7 therefore implies that the competing obstacle is strictly lower throughout (x, y) . By Step (i), $y \leq \zeta_i$ and both endpoints lie below $\overline{W}$; hence $\Psi_i \leq 0$ throughout (x, y) and payout is inactive. Thus the conglomerate continues on (x, y) . For $E := F - P_i > 0$,

$$\frac{\phi_c^2}{2} E'' + \gamma W E' - r E = -\Psi_i \geq 0, \quad E(x) = E(y) = 0. \quad (\text{D.23})$$

At an interior positive maximum, the left-hand side is strictly negative, a contradiction. Thus

each contact set is connected. Lemma 5 and $\eta_b > \eta_a$ imply that P_b is the upper obstacle near zero, while Step (i) gives $\Psi_b \leq 0$ up to the first positive P_b -contact. Applying the same maximum-principle argument between that contact point and $W = 0$, where $F(0) = P_b(0) = 0$, shows that a nonempty P_b -contact set must start at zero. Therefore

$$\mathbf{S}_a = (0, \widehat{W}_1] \quad \text{or} \quad \emptyset, \quad \mathbf{S}_b = [\widehat{W}_2, \widehat{W}_3] \quad \text{or} \quad \emptyset, \quad (\text{D.24})$$

where the second interval may be degenerate.

(iii) We prove the five-region upper bound and the ordering of the regions. If the stand-alone values do not cross, only P_b can bind, and Steps (i)–(ii) make the upper bound immediate. Suppose now that they cross transversely at W^* . Lemma 7 gives $P_b > P_a$ below W^* and $P_a > P_b$ above it, so P_b -contact can occur only below W^* and P_a -contact only above it. Moreover, $P'_b(W^*) < P'_a(W^*)$. Hence two nonempty spinoff intervals are separated by a nonempty continuation interval. Their upper endpoints lie strictly below $\overline{W}$, so the last spinoff interval is followed by another nonempty continuation interval before the terminal payout region $[\overline{W}, \infty)$ from Theorem 2. The only possible order is therefore

$$\mathbf{S}_a, \quad \mathbf{I}_1, \quad \mathbf{S}_b, \quad \mathbf{I}_2, \quad \mathbf{P}, \quad (\text{D.25})$$

with empty or degenerate blocks deleted. This proves the upper bound of five and the exact ordering when both spinoff sets have nonempty interiors.

D.2 Configurations with Fewer than Five Regions

Proposition 7 is a statement about the topology of the actual solution, not a verification of a conjectured four-boundary candidate. The rankings alone do not force all five regions to be nonempty: even when the stand-alone values cross once, the option to retain division a can remain strictly dominated by either continued conglomerate operation or retention of division b . Figure 9 depicts the maximal case in which both spinoff sets have nonempty interiors.

Corollary 7 (Fewer than five regions). *Suppose $\eta_b > \eta_a$ and $\mu_b < \mu_a$, and let $\mu_c = \mu_a + \mu_b - \kappa > \max\{\mu_a, \mu_b\}$, but drop the assumption of Proposition 7 that both divisions are spun off. Then $\mathbf{S}_a$ is either empty or an interval $(0, \widehat{W}_1]$ below the crossing point W^* of Lemma 7, and $\mathbf{S}_b$ is either empty or an interval $[\widehat{W}_2, \widehat{W}_3]$ above W^* and strictly below $\overline{W}$; $\mathbf{S}_b$ is empty if the stand-alone values do not cross. The optimal policy therefore consists of at most five connected regions, in the order of Proposition 7 with absent regions omitted. In particular, if the stand-alone values do not cross, or if they cross but division b is never spun off, the policy has at most three regions, $(0, \widehat{W}_1]$, $(\widehat{W}_1, \overline{W})$, and $[\overline{W}, \infty)$, and the conglomerate spins off division a*

under distress, as in the $\kappa = 0$ case of Proposition 6; and if neither division is ever spun off, the policy has two regions, $(0, \bar{W})$ and $[\bar{W}, \infty)$, and the conglomerate is liquidated as a whole.

D.3 An Additional Result: One Division Dominates

When the two rankings agree instead— $\mu_b \geq \mu_a$ and $\eta_b > \eta_a$, so division b dominates outright—the tension behind the five regions disappears and Section 5’s single-criterion ranking survives: $P(W; \mu, \eta)$ is increasing in both arguments (Lemma 3), so $P_b(W) \geq P_a(W)$, with equality only at $W = 0$.

Proposition 10. *The case with $\mu_b \geq \mu_a$ and $\eta_b > \eta_a$. Division b is never spun off. Whenever a spinoff occurs before liquidation, the division spun off is a .*

Proof of Proposition 10. Indeed, strict monotonicity in η and weak monotonicity in μ give $P_b(W) > P_a(W)$ for every $W > 0$. Since immediate spinoff of division a is feasible, $F(W) \geq P_b(W) > P_a(W)$, and therefore $\mathbf{S}_b = \{W > 0 : F(W) = P_a(W)\} = \emptyset$. The contact set $\mathbf{S}_a$ may be empty; if it is nonempty, its exercise action spins off division a . Figure 10 illustrates this conclusion in a parameterization with nonempty $\mathbf{S}_a$, so there is no conflict between the criteria. $\square$

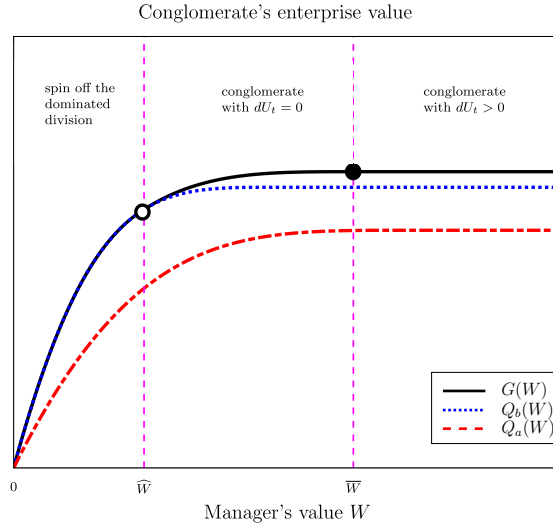


Figure 10: The conglomerate’s enterprise value under $\kappa > 0$, with $\eta_b > \eta_a$ and $\mu_b \geq \mu_a$: the dominated case. The solid line is $G(W)$; the dotted and dashed lines are $Q_b(W)$ and $Q_a(W)$. Division b dominates outright, $Q_b(W) = P_b(W) + W > P_a(W) + W = Q_a(W)$ for all $W > 0$, so only division a can be spun off. In the parameterization shown the contact set with Q_b is nonempty: the conglomerate spins off division a below $\bar{W}$ (open circle) and pays out cash above $\bar{W}$ (filled circle).

D.4 Proof of Proposition 8

We first consider the case $\kappa \geq \hat{\kappa}$. With running cost κ , the first-best value of the conglomerate firm equals

$$\frac{\mu_a + \mu_b - \kappa}{r}. \quad (\text{D.26})$$

If the running cost is sufficiently large so that

$$\frac{\mu_a + \mu_b - \kappa}{r} < \max_W P_a(W) + \max_W P_b(W), \quad (\text{D.27})$$

then even the first-best centralized value is below the decentralized value. Hence centralized financing cannot be optimal. The threshold $\hat{\kappa}$ is exactly the value of κ for which (D.27) holds with equality.

We next consider the case $\kappa \in (0, \hat{\kappa})$. Fix the stand-alone value functions and write

$$V_\kappa(\eta) := \max_{W \geq 0} F(W; \mu_c, \eta), \quad \mu_c = \mu_a + \mu_b - \kappa,$$

At fixed κ , increasing η reduces this volatility, so concavity of $F(W)$ and the comparison principle (Lemma 11) imply that V_κ is nondecreasing in η . As $\eta \rightarrow \infty$, $V_\kappa(\eta)$ converges to the first-best value $\mu_c/r > P_d^*$. Thus, defining the cutoff $\hat{\eta}_c$ as a function of κ as:

$$\hat{\eta}_c(\kappa) := \inf \{ \eta > 0 : V_\kappa(\eta) \geq P_d^* \}, \quad (\text{D.28})$$

we obtain a finite, uniquely defined lower boundary of the centralization region.

At fixed η , F is weakly increasing in μ_c . Hence an increase in κ , which reduces μ_c , weakly decreases F and therefore $V_\kappa(\eta)$. Since P_d^* is unchanged, $\hat{\eta}_c(\kappa)$ is nondecreasing.

Finally, suppose $\hat{\kappa} < \min\{\mu_a, \mu_b\}$, so the critical cost lies within the maintained parameter range. For every finite $\eta > 0$, we have the bounds $F(W; \mu_c, \eta) \leq \mu_c/r - W$ and $F(0; \mu_c, \eta) = 0$ imply $V_{\hat{\kappa}}(\eta) < \mu_c(\hat{\kappa})/r = P_d^*$. Thus $F(W; \mu_c, \eta)$ is strictly below μ_c/r at every $W \geq 0$. By continuity, $V_\kappa(\eta) < P_d^*$ for κ sufficiently close to $\hat{\kappa}$. Monotonicity in η then gives $\hat{\eta}_c(\kappa) > \eta$. Since η is arbitrary, the cutoff $\hat{\eta}_c$ diverges as $\kappa \uparrow \hat{\kappa}$.

D.5 Proof of Corollary 4

Write $\eta := \eta_a = \eta_b$. We first prove the dominance claim: strict monotonicity in μ at fixed η (Lemma 3) gives $P_a(W) > P_b(W)$ for every $W > 0$. Immediate spinoff of division b is feasible, so $F(W) \geq P_a(W) > P_b(W)$ and therefore $\mathbf{S}_a = \emptyset$.

For part (i), let $\tilde{P}_c(W) := P(W; \mu_c, \eta_c)$ denote the value of operating the conglomerate

without exercising either spinoff option, with $\mu_c = \mu_a + \mu_b - \kappa$, and suppose $\eta_c < \eta$. Lemma 5 gives $P'_a(0+) > \tilde{P}'_c(0+)$. Since both values vanish at zero, $P_a(W) > \tilde{P}_c(W)$ for all sufficiently small $W > 0$. If $\mathbf{S}_b$ were empty, the dominance claim would imply that neither spinoff obstacle binds anywhere on $(0, \infty)$. The variational inequality for F would then reduce to the no-spinoff problem, whose uniqueness follows from Lemma 11 with both obstacles deleted. Thus $F = \tilde{P}_c$, contradicting $F \geq P_a$ near zero. Hence $\mathbf{S}_b \neq \emptyset$. The argument in Appendix C.3 then applies with division a as the survivor. Lemma 6 holds because $\eta_c < \eta_a$ and $\mu_a < \mu_c$, so every contact point lies at or below $W_0 < \bar{W}_a$, and the maximum-principle step there shows that the contact set is a single interval, $\mathbf{S}_b = (0, \widehat{W}]$. Smooth pasting gives $F'(\widehat{W}) = P'_a(\widehat{W}) > -1$, so $\widehat{W} < \bar{W}$ and the region $\mathbf{I}$ is nonempty.

It remains to establish the equivalent running-cost condition in part (i). Equal risk-adjusted productivity gives $\phi_a = \mu_a/\eta > \phi_b = \mu_b/\eta$. Condition 2 implies $\mu_c > \mu_a$, so $\eta_c < \eta$ requires $\phi_c > \phi_a$. Cases a1 and a2 in Section 3.3 instead have $\phi_c \leq \phi_a$. Hence we are in case ab, where

$$\eta\phi_c = \sqrt{\mu_a^2 + \mu_b^2 + 2\rho\mu_a\mu_b}.$$

Thus $\eta_c < \eta$ is equivalent in this case to

$$\mu_a + \mu_b - \kappa < \sqrt{\mu_a^2 + \mu_b^2 + 2\rho\mu_a\mu_b},$$

which rearranges to the running-cost inequality in part (i).

Conversely, suppose the running-cost inequality in part (i) holds. Since $\mu_c > \mu_a > 0$, it implies

$$\mu_a^2 + \mu_b^2 + 2\rho\mu_a\mu_b > \mu_c^2 > \mu_a^2,$$

and therefore $\rho > -\mu_b/(2\mu_a) > -\mu_b/\mu_a = -\phi_b/\phi_a$. The parameters consequently lie in case ab, and the preceding formula for ϕ_c gives $\eta_c < \eta$. This establishes the equivalence in part (i) without an additional restriction on ρ .

For part (ii), suppose $\eta_c \geq \eta$. Since $\mu_c > \mu_a$, Lemma 3 implies

$$F(W) \geq \tilde{P}_c(W) > P_a(W), \quad W > 0.$$

Hence $\mathbf{S}_b = \emptyset$, including when $\eta_c = \eta$, and $\widehat{W} = 0$. This also proves the converse in part (i).

□

E Characterization of Value Functions

We first prove the one-project characterization in Lemma 9, including the regularity and bounds needed for the spinoff payoffs. We then prove Theorem 2 for the two-project problem with spinoff options. Finally, we apply the one-project argument to the baseline conglomerate without spinoff. Thus the properties of the stand-alone values are established before they are used in the proof for F .

E.1 The Single-Project Problem

Fix $\mu > 0$, $\sigma > 0$, $\lambda > 0$, and $\gamma > r > 0$, and put $\phi = \lambda\sigma$. Both the liquidation value and the manager's outside option are zero. Before termination, a one-project contract has dynamics

$$dW_t = \gamma W_t dt - dU_t + \alpha_t \sigma dB_t, \quad \alpha_t \geq \lambda, \quad \tau = \inf\{t \geq 0 : W_t = 0\}. \quad (\text{E.1})$$

Here B is a standard Brownian motion. The investor's value is

$$P(W) = \sup_{(\alpha, U)} \mathbb{E}_W \left[\int_0^\tau e^{-rt} (\mu dt - dU_t) \right]. \quad (\text{E.2})$$

An initial lump-sum payout is allowed, and paying the entire promise immediately terminates the contract.

Let $\Omega := (0, \infty)$ and $\bar{\Omega} := [0, \infty)$. We show that the investor's value function $P(W)$ solves the variational inequality

$$\max \left\{ \sup_{\alpha \geq \lambda} \left\{ \frac{1}{2} \alpha^2 \sigma^2 P''(W) \right\} + \gamma W P'(W) - r P(W) + \mu, \quad -1 - P'(W) \right\} = 0, \quad (\text{E.3})$$

with boundary condition

$$P(0) = 0. \quad (\text{E.4})$$

Because the promised utility must eventually be delivered to the manager and $\gamma > r$, payouts occur for sufficiently large W . Thus we impose the growth condition

$$\lim_{W \rightarrow \infty} \frac{P(W)}{W} = -1. \quad (\text{E.5})$$

Lemma 8 (Single-project value function). *The value function $P(W)$ is the unique viscosity solution to the variational inequality (E.3) on Ω , subject to the boundary condition (E.4) and*

the growth condition (E.5). Moreover:

1. **Concavity.** The value function $P(W)$ is concave. Because of this concavity, the maximizer in the continuation operator is $\alpha = \lambda$, and (E.3) reduces to

$$\max \left\{ \frac{1}{2} \lambda^2 \sigma^2 P''(W) + \gamma W P'(W) - r P(W) + \mu, \quad -1 - P'(W) \right\} = 0. \quad (\text{E.6})$$

2. **Free boundary.** There exists a unique payout boundary $\bar{W} \in (0, \frac{\mu}{\gamma-r})$ such that the payout region is

$$\mathbf{P} = [\bar{W}, \infty). \quad (\text{E.7})$$

On $[0, \bar{W})$, the value function is strictly concave.

3. **Smoothness and boundary conditions.** The function $P(W)$ is globally C^2 in $(0, \infty)$. Consequently,

$$P''(\bar{W}) = 0, \quad P'(\bar{W}) = -1, \quad P(\bar{W}) = \frac{\mu - \gamma \bar{W}}{r}. \quad (\text{E.8})$$

The value is also globally Lipschitz and bounded above, and belongs to $C^1([0, \infty))$, with right derivative at zero.

The optimal contract is thus characterized as follows.

Lemma 9. One-project setting. Under the optimal contract, the following results hold: (i) termination occurs at the first-hitting time $\tau := \inf\{t \geq 0 : W_t = 0\}$; (ii) only when the manager's promised value W_t is sufficiently high, i.e., exceeds the endogenous payout threshold $\bar{W}$, does the manager receive cash compensation $dU_t > 0$, given in (18); (iii) the optimal pay-for-performance sensitivity is $\beta_t^* = \lambda$ for all $t \leq \tau$; and (iv) the investor's value function $P(W)$ and the threshold $\bar{W}$ uniquely solve the ODE

$$rP(W) = \frac{\phi^2}{2} P''(W) + \gamma W P'(W) + \mu, \quad \forall W \in [0, \bar{W}], \quad (\text{E.9})$$

subject to (19), (20), (21), and (22).

Definition of viscosity solution.

Definition 1. A function $u(W)$ on $\bar{\Omega}$ is called a subsolution to (E.3) with boundary condition (E.4) and growth condition (E.5) if it satisfies:

- (1) u is upper semicontinuous on $\overline{\Omega}$.
(2) For any $g \in C^2(\overline{\Omega})$ such that $g - u$ attains a local minimum of 0 at some $W_0 \in \Omega$, we have

$$\max \left\{ \sup_{\alpha \geq \lambda} \left\{ \frac{1}{2} \alpha^2 \sigma^2 g''(W_0) \right\} + \gamma W_0 g'(W_0) - r g(W_0) + \mu, -1 - g'(W_0) \right\} \geq 0. \quad (\text{E.10})$$

- (3) $u(0) \leq 0$.
(4) $\limsup_{W \rightarrow +\infty} \frac{u(W)}{W} \leq -1$.

Definition 2. A function $v(W)$ on $\overline{\Omega}$ is called a supersolution to (E.3) with boundary condition (E.4) and growth condition (E.5) if it satisfies:

- (1) v is lower semicontinuous on $\overline{\Omega}$.
(2) For any $g \in C^2(\overline{\Omega})$ such that $g - v$ attains a local maximum of 0 at some $W_0 \in \Omega$, we have

$$\max \left\{ \sup_{\alpha \geq \lambda} \left\{ \frac{1}{2} \alpha^2 \sigma^2 g''(W_0) \right\} + \gamma W_0 g'(W_0) - r g(W_0) + \mu, -1 - g'(W_0) \right\} \leq 0. \quad (\text{E.11})$$

- (3) $v(0) \geq 0$.
(4) $\liminf_{W \rightarrow +\infty} \frac{v(W)}{W} \geq -1$.

Definition 3. A function is called a viscosity solution to (E.3) with boundary condition (E.4) and growth condition (E.5) if it is both a subsolution and a supersolution.

Proof of Lemma 8.

First, we prove that P is the unique viscosity solution. From classical textbooks such as Fleming and Soner (2006), the functions

$$P^*(W) := \limsup_{\zeta \rightarrow W} P(\zeta), \quad P_*(W) := \liminf_{\zeta \rightarrow W} P(\zeta) \quad (\text{E.12})$$

are respectively subsolution and supersolution by the dynamic programming principle.

We next show the comparison principle.

Lemma 10. If u and v are respectively a subsolution and a supersolution to (E.3) with the boundary condition (E.4) and the growth condition (E.5), then we must have $u \leq v$.

On the one hand, this lemma implies $P^*(W) \leq P_*(W)$. Since by definition $P^*(W) \geq P_*(W)$, we have $P(W) = P^*(W) = P_*(W)$. Thus, $P(W)$ is a viscosity solution. On the other hand, if there are two viscosity solutions $P_1(W)$, $P_2(W)$, then since P_1 is a subsolution while

P_2 is a supersolution, we have $P_1(W) \leq P_2(W)$. Similarly, we also have $P_1(W) \geq P_2(W)$. Therefore, $P_1(W) = P_2(W)$. That implies the uniqueness.

Proof of Lemma 10. We first make a small perturbation of the supersolution to justify the use of closed jets below. Fix $\epsilon_0 > 0$, put

$$\chi(W) := 1 - e^{-kW}, \quad v_{\epsilon_0}(W) := v(W) + \epsilon_0 \chi(W),$$

and choose $k > 0$ sufficiently large that

$$\frac{\lambda^2 \sigma^2}{2} \chi'' + \gamma W \chi' - r \chi < 0.$$

Such a choice is possible because, with $x = kW$,

$$e^{kW} \left(\frac{\lambda^2 \sigma^2}{2} \chi'' + \gamma W \chi' - r \chi \right) = -\frac{\lambda^2 \sigma^2 k^2}{2} + \gamma x - r(e^x - 1),$$

and the last two terms are bounded above for $x \geq 0$. If g touches v_{ϵ_0} from below, then $\tilde{g} := g - \epsilon_0 \chi$ touches v from below. The supersolution inequality, with the control supremum understood in the extended-real sense, forces $\tilde{g}'' \leq 0$. Consequently $g'' \leq \epsilon_0 \chi'' < 0$, and

$$\begin{aligned} \frac{\lambda^2 \sigma^2}{2} g'' + \gamma W g' - r g + \mu &= \frac{\lambda^2 \sigma^2}{2} \tilde{g}'' + \gamma W \tilde{g}' - r \tilde{g} + \mu + \epsilon_0 \left(\frac{\lambda^2 \sigma^2}{2} \chi'' + \gamma W \chi' - r \chi \right) < 0, \\ -1 - g' &\leq -1 - \tilde{g}' \leq 0. \end{aligned}$$

Thus v_{ϵ_0} is a supersolution with the same boundary inequality and growth condition. Moreover, every closed lower jet with curvature N at $W > 0$ satisfies $N \leq \epsilon_0 \chi''(W) < 0$. It suffices to prove $u \leq v_{\epsilon_0}$ for every $\epsilon_0 > 0$ and then let $\epsilon_0 \downarrow 0$. For the remainder of the comparison argument, write v for v_{ϵ_0} .

We prove by contradiction. If otherwise, choose $\widehat{W} \geq 0$ such that $u(\widehat{W}) - v(\widehat{W}) = 2\delta > 0$. Then, for $\xi > 1$ sufficiently close to 1, $\xi u(\widehat{W}) - v(\widehat{W}) > \delta > 0$; choose ξ also so that $(\xi - 1)\mu < r\delta/2$. The function $f(W) := \xi u(W)$ is a viscosity subsolution to

$$\max \left\{ \sup_{\alpha \geq \lambda} \left\{ \frac{1}{2} \alpha^2 \sigma^2 f'' \right\} + \gamma W f' - r f + \xi \mu, \quad -\xi - f' \right\} = 0, \quad (\text{E.13})$$

with the boundary inequality $f(0) \leq 0$ and growth condition

$$\limsup_{W \rightarrow +\infty} \frac{f(W)}{W} \leq -\xi. \quad (\text{E.14})$$

Consider

$$\varphi_m(W, \zeta) = f(W) - v(\zeta) - \frac{m}{2}(W - \zeta)^2, \quad (\text{E.15})$$

then $\max_{W, \zeta \geq 0} \varphi_m(W, \zeta) \geq \max_{W \geq 0} \{f(W) - v(W)\} > \delta > 0$. Moreover,

$$\arg \max_{W, \zeta \geq 0} \varphi_m(W, \zeta) \text{ exists and is uniformly bounded for } m \geq 1. \quad (\text{E.16})$$

By the growth condition, there are constants C_f, C_v such that

$$f(W) \leq -\frac{2\xi+1}{3}W + C_f, \quad v(W) \geq -\frac{\xi+2}{3}W + C_v, \quad \forall W \geq 0. \quad (\text{E.17})$$

Then

$$\varphi_m(W, \zeta) \quad (\text{E.18})$$

$$\leq -\frac{2\xi+1}{3}W + C_f + \frac{\xi+2}{3}\zeta - C_v - \frac{m}{2}(W - \zeta)^2 \quad (\text{E.19})$$

$$= -\frac{\xi-1}{3}W - \left[\frac{m}{2}(W - \zeta)^2 + \frac{\xi+2}{3}(W - \zeta) + \frac{(\xi+2)^2}{18m} \right] + C_f - C_v + \frac{(\xi+2)^2}{18m} \quad (\text{E.20})$$

$$= -\frac{\xi-1}{3}W - \left(\sqrt{\frac{m}{2}}(W - \zeta) + \frac{\xi+2}{3\sqrt{2m}} \right)^2 + C_f - C_v + \frac{(\xi+2)^2}{18m} \quad (\text{E.21})$$

$$\leq -\frac{\xi-1}{3}W + C_f - C_v + \frac{(\xi+2)^2}{18m}. \quad (\text{E.22})$$

Thus, any $W \geq \frac{3}{\xi-1} \left(C_f - C_v + \frac{(\xi+2)^2}{18m} \right)$ will result in a nonpositive $\varphi_m(W, \zeta)$, so we only need to focus on $\Omega_f := \left[0, \frac{3}{\xi-1} \left(C_f - C_v + \frac{(\xi+2)^2}{18m} \right) \right]$ for the optimal $\varphi_m(W, \zeta)$. Analogously, we also only need to focus on a bounded region Ω_ζ for ζ . Moreover, these two regions are uniformly bounded for any $m \geq 1$. Given the semicontinuity of f and v , we have (E.16).

For any $m = 1, 2, 3, \dots$, denote a maximizer by $(W_m, \zeta_m) \in \arg \max_{W, \zeta \geq 0} \varphi_m(W, \zeta)$. Since W_m, ζ_m are uniformly bounded, we can find a subsequence, which for notational simplicity is still denoted by (W_m, ζ_m) , which converges to (W_0, ζ_0) . Since $m \rightarrow +\infty$, we must have $W_0 = \zeta_0$. Semicontinuity and the positive gap give $f(W_0) - v(W_0) \geq \delta > 0$; since $f(0) \leq v(0)$, we have $W_0 = \zeta_0 > 0$. Thus both maximizing points are interior for all sufficiently large m . Therefore,

$$f(W_0) - v(W_0) = \varphi_m(W_0, W_0) \leq \varphi_m(W_m, \zeta_m) = f(W_m) - v(\zeta_m) - \frac{m}{2}(W_m - \zeta_m)^2. \quad (\text{E.23})$$

Taking $\limsup$ on the right hand side, we have

$$f(W_0) - v(W_0) \leq f(W_0) - v(W_0) - \liminf_{m \rightarrow +\infty} \frac{m}{2} (W_m - \zeta_m)^2. \quad (\text{E.24})$$

Thus $\liminf_{m \rightarrow +\infty} \frac{m}{2} (W_m - \zeta_m)^2 = 0$. By taking a subsequence, without loss of generality, we can assume

$$\lim_{m \rightarrow +\infty} \frac{m}{2} (W_m - \zeta_m)^2 = 0. \quad (\text{E.25})$$

In the following, for simplicity, we use (W, ζ) to denote (W_m, ζ_m) . According to Ishii's lemma, for any $\epsilon > 0$, there exist constants M, N in the corresponding closed upper and lower jets, with the matrix inequality (E.28) below. The perturbation gives $N \leq \epsilon_0 \chi''(\zeta) < 0$, and the matrix inequality gives $M \leq N$. Hence the control Hamiltonian is finite and continuous at both jet curvatures, so the viscosity inequalities pass to these closed jets and give

$$\max \left\{ \sup_{\alpha \geq \lambda} \left\{ \frac{1}{2} \alpha^2 \sigma^2 M \right\} + \gamma W \psi_W - r f + \xi \mu, \quad -\xi - \psi_W \right\} \geq 0, \quad (\text{E.26})$$

$$\max \left\{ \sup_{\alpha \geq \lambda} \left\{ \frac{1}{2} \alpha^2 \sigma^2 N \right\} + \gamma \zeta (-\psi_\zeta) - r v + \mu, \quad -1 + \psi_\zeta \right\} \leq 0, \quad (\text{E.27})$$

and

$$\begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \leq \nabla_{W, \zeta}^2 \psi + \epsilon (\nabla_{W, \zeta}^2 \psi)^2, \quad (\text{E.28})$$

where $\psi(W, \zeta) = \frac{m}{2} (W - \zeta)^2$. By direct calculation, we have

$$\psi_W = -\psi_\zeta = m(W - \zeta), \quad (\text{E.29})$$

and

$$\nabla_{W, \zeta}^2 \psi = \begin{pmatrix} m & -m \\ -m & m \end{pmatrix}.$$

From (E.27) we have $N \leq 0$. Then from (E.28), by setting $X = (1, 1)^\top$, we have

$$X^\top \begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} X \leq X^\top (\nabla_{W, \zeta}^2 \psi) X + \epsilon X^\top (\nabla_{W, \zeta}^2 \psi)^2 X. \quad (\text{E.30})$$

Since both terms on the right hand side vanish, we have

$$M - N \leq m(1 - 1)^2 = 0. \quad (\text{E.31})$$

That is $M \leq N \leq 0$. Therefore, (E.26) and (E.27) become

$$\max \left\{ \frac{1}{2} \lambda^2 \sigma^2 M + \gamma W \psi_W - r f + \xi \mu, \quad -\xi - \psi_W \right\} \geq 0, \quad (\text{E.32})$$

$$\max \left\{ \frac{1}{2} \lambda^2 \sigma^2 N + \gamma \zeta (-\psi_\zeta) - r v + \mu, \quad -1 + \psi_\zeta \right\} \leq 0. \quad (\text{E.33})$$

From (E.32), there are two cases:

1. $\psi_W \leq -\xi$.

From (E.33), we have

$$-1 \leq -\psi_\zeta = \psi_W \leq -\xi, \quad (\text{E.34})$$

which contradicts $\xi > 1$.

2. $\psi_W > -\xi$.

From (E.32) and (E.33) we have, by the initial choice of ξ and for sufficiently large m ,

$$0 \leq \frac{1}{2} \lambda^2 \sigma^2 M + \gamma W \psi_W - r f + \xi \mu - \left(\frac{1}{2} \lambda^2 \sigma^2 N + \gamma \zeta (-\psi_\zeta) - r v + \mu \right) \quad (\text{E.35})$$

$$= \frac{1}{2} \lambda^2 \sigma^2 (M - N) + \gamma m (W - \zeta)^2 - r(f - v) + (\xi - 1)\mu \quad (\text{E.36})$$

$$\leq o(1) - r\delta + (\xi - 1)\mu \quad (\text{E.37})$$

$$< 0. \quad (\text{E.38})$$

That is a contradiction. Returning to the original notation, $u \leq v_{\epsilon_0}$ for every $\epsilon_0 > 0$. Letting $\epsilon_0 \downarrow 0$ proves $u \leq v$. $\square$

Second, we show that P is concave. We prove by contradiction. Otherwise, assume that $P\left(\frac{W_1 + W_2}{2}\right) < \frac{1}{2} \left(P(W_1) + P(W_2) \right)$ for $0 < W_1 < W_2$. Then we choose a constant $M \geq \lambda$, and set $\alpha = M$ for $W \in [W_1, W_2]$, with no payouts before exit. At each endpoint we concatenate an ε -optimal contract; boundedness of the state before exit and the stopped Itô formula for $e^{-\gamma t} W_t$ verify promise keeping. By the dynamic programming principle, dropping the nonnegative running payoff and then letting $\varepsilon \downarrow 0$, we have

$$P\left(\frac{W_1 + W_2}{2}\right) \geq \mathbb{E} \left[e^{-r\tau^M} \left\{ P(W_1) \mathbf{1}_{\{\tau_1^M < \tau_2^M\}} + P(W_2) \mathbf{1}_{\{\tau_2^M < \tau_1^M\}} \right\} \right], \quad (\text{E.39})$$

where $\tau^M = \tau_1^M \wedge \tau_2^M$ and

$$\tau_1^M = \inf\{t \geq 0 | W_t^M = W_1\}, \quad \tau_2^M = \inf\{t \geq 0 | W_t^M = W_2\}, \quad (\text{E.40})$$

and

$$dW_t^M = \gamma W_t^M dt + \sigma M dZ_t, \quad W_0^M = \frac{W_1 + W_2}{2}. \quad (\text{E.41})$$

For $q(w) = (w - W_1)(W_2 - w)$, the generator before exit is

$$\gamma w q'(w) + \frac{\sigma^2 M^2}{2} q''(w) = \gamma w (W_1 + W_2 - 2w) - \sigma^2 M^2 \leq -\frac{\sigma^2 M^2}{2}$$

for sufficiently large M . Stopped Itô estimates therefore give $\mathbb{E}[\tau^M] = O(M^{-2})$. Also,

$$\mathbb{E}[W_{\tau^M}^M] = \frac{W_1 + W_2}{2} + \gamma \mathbb{E} \int_0^{\tau^M} W_s^M ds \longrightarrow \frac{W_1 + W_2}{2}.$$

As $M \rightarrow \infty$, the first exit time τ^M tends to zero in L^1 , and the two exit probabilities converge to $1/2$. Moreover, $e^{-r\tau^M} \rightarrow 1$ in L^1 . Therefore, from (E.39), we have

$$P\left(\frac{W_1 + W_2}{2}\right) \geq \frac{1}{2} \left(P(W_1) + P(W_2) \right). \quad (\text{E.42})$$

That is a contradiction. The bounds below give local boundedness, so midpoint concavity implies concavity and continuity in the interior. Continuity at zero then extends concavity to $[0, \infty)$.

Third, we prove the existence of the free boundary. By concavity, the right derivative $P'_+(W)$ exists and is nonincreasing for $W > 0$. The admissibility of immediate payouts gives $P(W + h) \geq P(W) - h$ for $h > 0$, so $P'_+(W) \geq -1$. Thus, if $P'_+(W_0) = -1$ for some $W_0 > 0$, then $P'_+(W) = -1$ for every $W \geq W_0$. Define

$$\overline{W} := \inf\{W > 0 : P'_+(W) = -1\}, \quad \inf \emptyset := \infty.$$

Then $P'_+(W) > -1$ for $0 < W < \overline{W}$. If $\overline{W} < \infty$, P is linear with slope -1 above $\overline{W}$; by continuity, $P(W) = P(\overline{W}) - (W - \overline{W})$ for $W \geq \overline{W}$. This establishes the threshold structure, allowing at this stage $\overline{W} = \infty$, in which case the payout region is empty.

Fourth, we show that P is globally C^2 in $(0, \infty)$ by the following three steps.

1). P is C^2 in $(0, \overline{W})$.

On any compact interval $[W_1, W_2] \subset (0, \overline{W})$, $P(W)$ is a viscosity solution to the linear

second order operator

$$\frac{1}{2}\lambda^2\sigma^2 f'' + \gamma W f' - r f + \mu = 0 \quad (\text{E.43})$$

with boundary conditions

$$f(W_1) = P(W_1), \quad f(W_2) = P(W_2). \quad (\text{E.44})$$

On the one hand, according to Crandall, Ishii, and Lions (1992), the above PDE problem allows a unique viscosity solution, which must be $P(W)$. On the other hand, the above ODE problem allows a unique classical solution. Since the classical solution must be a viscosity solution, $P(W)$ is also a classical solution, and it is C^2 in $(0, \bar{W})$.

2). If $\bar{W} < \infty$, P is C^2 in $(\bar{W}, \infty)$.

Since $P' = -1$ on this interval, P is linear and thus C^2 .

3). If $0 < \bar{W} < \infty$, $P'_-(\bar{W}) = P'_+(\bar{W}) = -1$, $P''_-(\bar{W}) = P''_+(\bar{W}) = 0$.

We prove by contradiction. From the concavity we have the existence of left derivative. If $P'_-(\bar{W}) \neq -1$, then by the concavity of P , we have $P'_-(\bar{W}) = -1 + \epsilon > -1$. For any $M > 0$, the function

$$f(W) := P(\bar{W}) + (-1 + \frac{\epsilon}{2})(W - \bar{W}) - \frac{1}{2}M^2(W - \bar{W})^2 \quad (\text{E.45})$$

is locally larger than $P(W)$ around $\bar{W}$. By definition of viscosity subsolution,

$$0 \leq \max \left\{ \frac{1}{2}\lambda^2\sigma^2 f'' + \gamma W f' - r f + \mu, \quad -1 - f' \right\} \quad (\text{E.46})$$

$$= \max \left\{ -\frac{1}{2}\lambda^2\sigma^2 M^2 + \gamma W(-1 + \frac{\epsilon}{2}) - r P(\bar{W}) + \mu, \quad -\frac{\epsilon}{2} \right\} < 0 \quad (\text{E.47})$$

when M is large enough. That is a contradiction.

If $P''_-(\bar{W}) \neq 0$, then by the concavity, there should be a sequence $W_k \rightarrow \bar{W}-$, $k = 1, 2, 3, \dots$, such that

$$\lim_{k \rightarrow +\infty} P''(W_k) = -\epsilon < 0. \quad (\text{E.48})$$

Therefore, from the second order operator,

$$\gamma W_k P'(W_k) - r P(W_k) + \mu = -\frac{1}{2}\lambda^2\sigma^2 P''(W_k) \quad (\text{E.49})$$

By taking $k \rightarrow +\infty$, we have

$$-\gamma\overline{W} - rP(\overline{W}) + \mu = \frac{1}{2}\lambda^2\sigma^2\epsilon > 0. \quad (\text{E.50})$$

Therefore, for $W > \overline{W}$ and sufficiently close to $\overline{W}$, we have

$$\max \left\{ \frac{1}{2}\lambda^2\sigma^2P'' + \gamma WP' - rP + \mu, \quad -1 - P' \right\} \quad (\text{E.51})$$

$$\geq \frac{1}{2}\lambda^2\sigma^2P'' + \gamma WP' - rP + \mu \quad (\text{E.52})$$

$$= -\gamma W - rP(W) + \mu \quad (\text{E.53})$$

$$> 0. \quad (\text{E.54})$$

That is a contradiction.

Fifth, we prove that $0 < \overline{W} < \mu/(\gamma - r)$.

We first show that $0 < \overline{W} < \infty$.

1. If $\overline{W} = 0$, then for any $W \geq 0$, $P(W) = -W$, $P'(W) = -1$, and $P''(W) = 0$. Thus, near $W = 0$, P does not satisfy the variational inequality (E.3). It is a contradiction.

2. If $\overline{W} = \infty$, the payout branch is strict throughout $(0, \infty)$, and the preceding regularity argument gives the classical continuation ODE on this entire interval.

By the concavity and the growth condition, $\lim_{W \rightarrow \infty} P'(W) = -1$. Because $P'' \leq 0$ and $\int_a^\infty (-P''(W)) dW = P'(a) + 1 < \infty$ for any $a > 0$, along a sequence of $W_k \rightarrow \infty$, $k \geq 1$, we have

$$P''(W_k) \rightarrow 0, \quad P'(W_k) \rightarrow -1, \quad \frac{P(W_k)}{W_k} \rightarrow -1. \quad (\text{E.55})$$

Since $\overline{W} = \infty$, we have for $W = W_k$,

$$0 = \sup_{\alpha \geq \lambda} \left\{ \frac{1}{2}\alpha^2\sigma^2P'' \right\} + \gamma W_k P' - rP + \mu \quad (\text{E.56})$$

$$= \frac{1}{2}\lambda^2\sigma^2P'' + \gamma W_k P' - rP + \mu \quad (\text{E.57})$$

$$= -(\gamma + o(1))W_k + (r + o(1))W_k + \mu + o(1) \quad (\text{E.58})$$

$$= -(\gamma - r + o(1))W_k \quad (\text{E.59})$$

$$< 0. \quad (\text{E.60})$$

Contradiction.

It remains to prove the upper bound $\bar{W} < \mu/(\gamma - r)$. Since P is C^2 , we have at $\bar{W}-$,

$$0 = \frac{1}{2}\lambda^2\sigma^2P'' + \gamma\bar{W}P' - rP + \mu = -\gamma\bar{W} - rP + \mu. \quad (\text{E.61})$$

Given $P' > -1$ on $(0, \bar{W})$ and $P(0) = 0$, integration gives $P(\bar{W}) > -\bar{W}$. Therefore, the right hand side is strictly less than $-(\gamma - r)\bar{W} + \mu$. Since the right hand side equals zero, $0 < -(\gamma - r)\bar{W} + \mu$, and that finishes our proof.

Finally, we show $P(W)$ is strictly concave in $[0, \bar{W}]$. We prove by contradiction. Otherwise, assume $P''(W) = 0$ at some $W \in (0, \bar{W})$, then from the variational inequality, we have

$$\gamma WP'(W) - rP(W) + \mu = 0.$$

Promise keeping and $\gamma > r$ imply that the investor's discounted compensation cost is at least W , while discounted gross output is at most μ/r . Hence $P(W) \leq \mu/r - W$, which gives

$$\gamma WP'(W) = rP(W) - \mu \leq -rW, \quad P'(W) \leq -\frac{r}{\gamma} < 0.$$

Given $P''(W) = 0$, for small $\Delta > 0$ such that $W + \Delta < \bar{W}$,

$$\gamma(W + \Delta)P'(W + \Delta) - rP(W + \Delta) + \mu \quad (\text{E.62})$$

$$= (\gamma - r)\Delta P'(W) + o(\Delta) \quad (\text{E.63})$$

$$< 0. \quad (\text{E.64})$$

From the variational inequality, that implies $P''(W + \Delta) > 0$, contradiction to the concavity. At zero, the continuation ODE gives $P''(0+) = -2\mu/(\lambda^2\sigma^2) < 0$. Thus $P'' < 0$ below the payout boundary, which proves the stated strict concavity. $\square$

E.2 The Two-Project Problem with Spinoff Options

E.2.1 Viscosity Solutions and Comparison

An upper-semicontinuous function u is a viscosity subsolution if, whenever $g - u$ has a local minimum of zero at $W_0 \in \Omega$,

$$\max \{\mathcal{L}g(W_0), -1 - g'(W_0), P_a(W_0) - g(W_0), P_b(W_0) - g(W_0)\} \geq 0, \quad (\text{E.65})$$

and if $u(0) \leq 0$ and $\limsup_{W \rightarrow \infty} u(W)/W \leq -1$. A lower-semicontinuous function v is a viscosity supersolution if, whenever $g - v$ has a local maximum of zero at $W_0 \in \Omega$, the same maximum in (E.65), evaluated at the lower test function $g \in C^2$, is no greater than zero, and

if $v(0) \geq 0$ and $\liminf_{W \rightarrow \infty} v(W)/W \geq -1$. A continuous function that is both a subsolution and a supersolution is a viscosity solution.

Lemma 11 (Comparison principle). *If u and v are, respectively, a viscosity subsolution and a viscosity supersolution of (42), with the boundary and growth inequalities associated with (44) and (45), then $u \leq v$ on $\bar{\Omega}$. The same conclusion holds after deleting either or both spinoff obstacles.*

Proof of Lemma 11. We extend the scalar comparison argument in Lemma 10, replacing (μ, ϕ) by (μ_c, ϕ_c) . Let L be a common global Lipschitz constant of P_a and P_b , and choose

$$K = 1 + \frac{\mu_c}{r} + \max\{0, \sup_{W \geq 0} P_a(W)\} + \max\{0, \sup_{W \geq 0} P_b(W)\}.$$

Use the same strict concave perturbation $v_\varepsilon = v + \varepsilon(1 - e^{-\alpha W})$. The scalar proof shows that its continuation and gradient inequalities hold. Its obstacle inequalities also hold because the perturbation is nonnegative.

Suppose that $u - v_\varepsilon$ is positive somewhere. Choose $\delta > 0$ and $\xi > 1$ close enough to one that $\sup(\xi u - v_\varepsilon) > 2\delta$, $(\xi - 1)K < \delta/2$, and $(\xi - 1)\mu_c < r\delta/2$. Set $f = \xi u$. The scaled subsolution has payout branch $-\xi - f'$ and obstacle branches $\xi P_i - f$. The compactness, doubling, and strictly negative closed-jet curvature arguments from the scalar proof apply unchanged. They give maximizing pairs (W_m, Z_m) , a common slope p_m , and curvatures $M_m \leq N_m < 0$, with

$$m(W_m - Z_m)^2 \rightarrow 0, \quad f(W_m) - v_\varepsilon(Z_m) \geq \delta.$$

The supersolution inequalities give $p_m \geq -1$ and $v_\varepsilon(Z_m) \geq P_i(Z_m)$. Thus the scaled payout branch is strictly negative. If a scaled obstacle branch were nonnegative, then

$$\begin{aligned} \delta &\leq f(W_m) - v_\varepsilon(Z_m) \leq \xi P_i(W_m) - P_i(Z_m) \\ &\leq (\xi - 1)K + L|W_m - Z_m| < \delta \end{aligned}$$

for large m , a contradiction. Both obstacle branches are therefore strict. The continuation branches give, exactly as in the scalar proof,

$$\begin{aligned} 0 &\leq \frac{\phi_c^2}{2}(M_m - N_m) + \gamma m(W_m - Z_m)^2 - r\{f(W_m) - v_\varepsilon(Z_m)\} + (\xi - 1)\mu_c \\ &\leq o(1) - r\delta + (\xi - 1)\mu_c < 0. \end{aligned}$$

Thus $u \leq v_\varepsilon$; letting $\varepsilon \downarrow 0$ proves $u \leq v$. Deleting either or both obstacles only removes the corresponding branch checks. $\square$

The same comparison argument gives monotonicity in the running and stopping payoffs. Monotonicity in effective continuation variance uses concavity.

E.2.2 Proof of Theorem 2

The proof proceeds in five steps.

(i) We prove concavity and identify the optimal sensitivity pair.

Apply the high-volatility exit argument in the proof of Lemma 8. Fix $0 < W_1 < W_2$, $\theta \in (0, 1)$, and $W_\theta = \theta W_1 + (1 - \theta)W_2$. Choose $\beta^0 \in \mathcal{C}$ and use $\beta^M = M\beta^0 \in \mathcal{C}$, with no payouts or spinoffs, until the first exit τ_M from (W_1, W_2) . Concatenate ε -optimal conglomerate contracts at the two endpoints. The weak DPP gives

$$F(W_\theta) \geq \mathbb{E} \left[\int_0^{\tau_M} e^{-rt} \mu_c dt + e^{-r\tau_M} \{F(W_{\tau_M}) - \varepsilon\} \right]. \quad (\text{E.66})$$

Since $h(\beta^M) = M^2 h(\beta^0)$, the scalar proof's stopped Itô estimates apply with $\sigma^2 = h(\beta^0)$. They give $\mathbb{E}[\tau_M] = O(M^{-2})$, endpoint probabilities converging to θ and $1 - \theta$, and $e^{-r\tau_M} \rightarrow 1$ in L^1 . Letting $M \rightarrow \infty$ and then $\varepsilon \downarrow 0$ yields

$$F(W_\theta) \geq \theta F(W_1) + (1 - \theta)F(W_2).$$

Thus F is concave. As in the scalar proof, only ε -optimal endpoint contracts are used; existence of an optimizer is not assumed.

Let $I \subset \Omega$ be an open interval on which both spinoff obstacles and the payout branch are strict. We first show that F is a viscosity solution on I of the uniformly elliptic linear equation

$$\frac{\phi_c^2}{2} F''(W) + \gamma W F'(W) - rF(W) + \mu_c = 0. \quad (\text{E.67})$$

Standard one-dimensional interior regularity for this uniformly elliptic linear equation yields $F \in C^2(I)$, and the equation holds classically. Concavity then gives $F'' \leq 0$ pointwise, so the continuation Hamiltonian is attained at the unique minimizer of h . Strict convexity of h makes (β_a^*, β_b^*) the unique optimal pair wherever $F'' < 0$; if $F'' = 0$, the sensitivity choice does not affect value.

(ii) We prove existence of the payout boundary $\overline{W}$ and $0 < \overline{W} < \infty$.

An immediate payout of $W - x$ is feasible whenever $0 \leq x < W$. Therefore,

$$F(W) \geq F(x) - (W - x), \quad 0 \leq x < W, \quad (\text{E.68})$$

so $G(W) := F(W) + W$ is nondecreasing. Equivalently, every one-sided derivative of F is at

least -1 . Since F is concave, its one-sided derivatives are nonincreasing. Once a derivative reaches -1 , all subsequent derivatives equal -1 . Thus the payout set is an upper interval $[\overline{W}, \infty)$ and

$$F(W) = F(\overline{W}) - (W - \overline{W}), \quad W \geq \overline{W}, \quad (\text{E.69})$$

provided that $0 < \overline{W} < \infty$.

If $\overline{W} = 0$, then $F(W) = -W$. For all sufficiently small $W > 0$, however,

$$\mathcal{L}F(W) = \mu_c - (\gamma - r)W > 0, \quad (\text{E.70})$$

which violates (42). Hence $\overline{W} > 0$.

Suppose next that $\overline{W} = \infty$. Let $W_0 > \max\{\overline{W}_a, \overline{W}_b\}$. For $W \geq W_0$, each obstacle is affine with slope -1 . If F touched either obstacle there, (E.68), concavity, and tangency to that affine obstacle would force the slope of F to equal -1 , contrary to $\overline{W} = \infty$. Hence both obstacles are strict on this tail. The gradient constraint is also strict at every finite tail point under the supposition $\overline{W} = \infty$; one-dimensional elliptic regularity therefore makes the tail a classical continuation interval on which (E.67) holds. Concavity and the growth condition imply

$$F'(W) \longrightarrow -1, \quad \frac{F(W)}{W} \longrightarrow -1. \quad (\text{E.71})$$

Moreover, because F' is monotone and has finite total variation on the tail, there is a sequence $W_k \rightarrow \infty$ for which $F''(W_k) \rightarrow 0$.

Dividing (E.67) by W_k and passing to the limit gives

$$0 = -\gamma + r, \quad (\text{E.72})$$

contrary to $\gamma > r$. Therefore $\overline{W} < \infty$.

(iii) We prove C^1 fitting at the spinoff boundaries. Moreover, the spinoff region $\mathbf{S}_i \subset [0, \min\{\overline{W}, \overline{W}_{-i}\})$.

First, we prove the C^1 fit. The obstacle inequalities give

$$F(W) \geq P_i(W), \quad i = a, b. \quad (\text{E.73})$$

Suppose that $F(W_0) = P_i(W_0)$ at an interior point. Since $F - P_i$ has a local minimum of zero

and P_i is continuously differentiable,

$$F'_-(W_0) \leq P'_i(W_0) \leq F'_+(W_0). \quad (\text{E.74})$$

Concavity gives the reverse ordering $F'_-(W_0) \geq F'_+(W_0)$. Hence

$$F'_-(W_0) = F'_+(W_0) = P'_i(W_0). \quad (\text{E.75})$$

If both obstacles bind, the same argument shows that their derivatives agree with the derivative of F . Thus F satisfies value matching and smooth fit at every interior spinoff boundary.

Next, we prove that the spinoff region $\mathbf{S}_i \subset [0, \overline{W}_{-i})$. At the stand-alone payout boundary,

$$\mathcal{L}P_i(\overline{W}_i) = \mu_c - \mu_i. \quad (\text{E.76})$$

Since $\mu_c > \mu_i$, continuing the integrated firm for a short time and then spinning off strictly dominates immediate spinoff at $\overline{W}_i$. Thus $F(\overline{W}_i) > P_i(\overline{W}_i)$. Since $F + W$ is nondecreasing while $P_i + W$ is constant above $\overline{W}_i$, it follows that

$$F(W) > P_i(W), \quad W \geq \overline{W}_i. \quad (\text{E.77})$$

Consequently, under strict productivity superiority, every contact with P_i lies below its stand-alone payout boundary.

Finally, we prove that the spinoff region $\mathbf{S}_i \subset [0, \overline{W})$. Otherwise $G = F + W$ would be constant to its right, while $Q_i = P_i + W$ is strictly increasing below $\overline{W}_i$, so Q_i would immediately overtake G , contradicting $F \geq P_i$.

(iv) We prove smooth pasting and super-contact (C^2 fitting) at the payout boundary. Moreover, $F \in C_{\text{loc}}^{1,1}(\Omega)$.

We first prove the C^2 fitting. By (E.69), $F'(\overline{W}+) = -1$. By Step (iii), both spinoff obstacles are strict at $\overline{W}$. If $F'(\overline{W}-) = -1 + \varepsilon$ for some $\varepsilon > 0$, choose a quadratic test function that touches F from above at $\overline{W}$, has slope $-1 + \varepsilon/2$, and has an arbitrarily large negative second derivative. The obstacle branches and payout branch are then strictly negative, and the continuation branch is also negative when the curvature is sufficiently large. This contradicts the subsolution inequality. Therefore,

$$F'(\overline{W}-) = F'(\overline{W}+) = -1. \quad (\text{E.78})$$

The continuation ODE holds on a left neighborhood of $\overline{W}$. It implies that the left limit of F'' exists. If this limit were negative, then, as in the proof of Lemma 8, the continuation

residual evaluated on the linear tail would be strictly positive immediately to the right of $\overline{W}$, contradicting (42). Thus

$$F''(\overline{W}-) = F''(\overline{W}+) = 0. \quad (\text{E.79})$$

We next prove $F \in C_{\text{loc}}^{1,1}(\Omega)$. The applicable one-dimensional estimate is the following. Let $J \Subset (0, \infty)$, let $a > 0$, let b and f be locally Lipschitz, and let $\psi \in C^{1,1}(J)$. If a concave continuous viscosity solution u satisfies

$$\max \{au'' + b(W)u' - ru + f(W), \psi - u, -1 - u'\} = 0 \quad \text{on } J,$$

then $u \in C_{\text{loc}}^{1,1}(J)$. To obtain the local estimate, fix $J' \Subset J$ and apply the viscosity doubling argument to the second difference $u(W+h) + u(W-h) - 2u(W)$. The uniformly elliptic continuation branch bounds this difference below by $-Ch^2$; the obstacle branch supplies the same bound from the $C^{1,1}$ norm of ψ ; and the gradient-constraint branch is affine. Concavity supplies the upper bound zero. Thus

$$-Ch^2 \leq u(W+h) + u(W-h) - 2u(W) \leq 0$$

whenever $W, W \pm h \in J'$. The difference-quotient characterization of $C^{1,1}$ then gives the claim. The constant C depends only on the ellipticity constant and the local coefficient, obstacle, and solution bounds.

Here concavity reduces the continuation operator to

$$\mathcal{A}_c f = \frac{\phi_c^2}{2} f'' + \gamma W f' - r f + \mu_c, \quad \phi_c > 0.$$

Each P_i is locally $C^{1,1}$. Put $\psi := \max\{P_a, P_b\}$. If $P_a = P_b$ at a transverse crossing, contact there is impossible: the derivative squeeze in Step (iii) would force $P'_a = P'_b$. Hence the obstacle is inactive in a neighborhood of such a crossing. At every possible contact, either one obstacle is locally effective or the tied obstacles have the same first derivative; in the latter case their maximum is locally $C^{1,1}$. Thus, on each compact subinterval, the obstacle can be replaced away from the contact set by a $C^{1,1}$ function without changing the variational inequality. The preceding estimate applies and gives

$$F \in C_{\text{loc}}^{1,1}(\Omega).$$

In particular, F' is locally absolutely continuous, F'' exists almost everywhere and is locally bounded.

(v) We verify the optimal contract.

For a fixed admissible sensitivity pair β_t , define

$$\mathcal{L}^{\beta_t} F = \frac{h(\beta_t)}{2} F'' + \gamma W F' - r F + \mu_c. \quad (\text{E.80})$$

Let τ_* be the first time integrated operation ends, either through a spinoff or at $W = 0$. An admissible pre-spinoff control consists of a progressively measurable process $\beta_t \in \mathcal{C}$ and an adapted, nondecreasing, càdlàg, locally finite process U such that

$$W_t \geq 0 \quad (t < \tau_*), \quad 0 \leq \Delta U_t \leq W_{t-}.$$

The time τ_* is a stopping time, and the choice of the division spun off at τ_* is $\mathcal{F}_{\tau_*}$ -measurable. We also impose

$$\int_0^{T \wedge \tau_*} h(\beta_s) ds < \infty \quad \text{a.s. for every } T < \infty, \quad \mathbb{E} \left[\int_0^\infty e^{-rs} dU_s^{\text{tot}} \right] < \infty, \quad (\text{E.81})$$

where U^{tot} includes the manager's compensation after a spinoff. The second restriction is without loss because a contract violating it has infinite discounted compensation cost, whereas discounted gross output is bounded above.

For $n \geq 1$, define

$$\theta_{n,T} := T \wedge \tau_* \wedge \inf\{t : W_t \notin (1/n, n)\} \wedge \inf\left\{t : \int_0^t h(\beta_s) ds \geq n\right\} \wedge \inf\{t : U_t \geq n\}.$$

Because F' is bounded on $[1/n, n]$, the martingale term in the Itô formula below, stopped at $\theta_{n,T}$, is square-integrable and has expectation zero.

Promise keeping gives, for every stopping time $\theta \leq \tau_*$,

$$\begin{aligned} e^{-r\theta} W_\theta &= \mathbb{E}_\theta \left[\int_\theta^\infty e^{-r\theta - \gamma(s-\theta)} dU_s^{\text{tot}} \right] \\ &\leq \mathbb{E}_\theta \left[\int_\theta^\infty e^{-rs} dU_s^{\text{tot}} \right]. \end{aligned}$$

The right-hand side is bounded by a conditional expectation of the integrable random variable $\int_0^\infty e^{-rs} dU_s^{\text{tot}}$. Hence the localized terminal promised utilities are uniformly integrable. Together with

$$|F(W)| \leq W + \frac{\mu_a + \mu_b}{r},$$

this justifies passage first through $\theta_{n,T} \uparrow T \wedge \tau_*$ and then through $T \uparrow \infty$.

Let U^c denote the continuous part of U . For any bounded localization time $\theta \leq \tau_*$, the generalized Itô–Tanaka formula gives

$$\begin{aligned}
& e^{-r\theta} F(W_\theta) - F(W_0) + \int_0^\theta e^{-rs} (\mu_c ds - dU_s) \\
&= \int_0^\theta e^{-rs} [\mathcal{L}^{\beta_s} F(W_{s-}) ds - (1 + F'(W_{s-})) dU_s^c] + M_\theta \\
&+ \sum_{0 < s \leq \theta} e^{-rs} [F(W_{s-} - \Delta U_s) - F(W_{s-}) - \Delta U_s]. \tag{E.82}
\end{aligned}$$

Local $C^{1,1}$ regularity makes F' locally absolutely continuous, so its almost-everywhere second derivative enters this formula without additional local-time terms. By (E.68), every term in the jump sum is nonpositive. Under the feedback policy below it is zero, because any positive-time payout jump moves promised utility only through the linear payout tail. If $W_0 > \bar{W}$, first apply (E.68) to the initial lump-sum reduction to $\bar{W}$, and then apply (E.82) from the post-payment state; equality holds because this reduction also lies entirely in the linear tail.

At every point where F'' exists, concavity and the variational inequality imply $\mathcal{L}^\beta F \leq 0$ for every admissible β ; also $1 + F' \geq 0$ and $F \geq P_a, P_b$. The conditional promise-keeping estimate above, specialized to deterministic T on $\{T < \tau_*\}$, gives

$$\mathbb{E} [e^{-rT} W_T \mathbf{1}_{\{T < \tau_*\}}] \leq \mathbb{E} \left[\int_T^\infty e^{-rs} dU_s^{\text{tot}} \right] \longrightarrow 0. \tag{E.83}$$

The growth bound then makes $e^{-rT} F(W_T) \mathbf{1}_{\{T < \tau_*\}}$ uniformly integrable and forces its expectation to zero. Apply (E.82) at $\theta_{n,T}$. The stopped martingale has mean zero; the drift and payout terms are nonpositive; and at τ_* the terminal value dominates the appropriate spinoff payoff, or equals zero at liquidation. Uniform integrability permits $n \rightarrow \infty$, and (E.83) permits $T \rightarrow \infty$. Thus $F(W_0)$ is an upper bound on the payoff of every admissible contract.

Now use (β_a^*, β_b^*) while the firm is in the integrated continuation region, make payouts only at $\bar{W}$ through the minimal process that reflects W_t downward there, and stop integrated operation at the first contact with either spinoff obstacle. If $F = P_a$, spin off project b and use the optimal stand-alone contract for project a ; if $F = P_b$, spin off project a and use the optimal stand-alone contract for project b . At a tie, either value-matching action is optimal. Terminate when W_t reaches zero.

It remains to verify that this feedback control is admissible. Let

$$\mathcal{S} := \{0\} \cup \{W : F(W) = P_a(W)\} \cup \{W : F(W) = P_b(W)\}.$$

After any initial lump-sum reduction from $W_0 > \bar{W}$ to $\bar{W}$, write $w_0 := W_0 \wedge \bar{W}$ for the

post-payment state. Until its first entrance into $\mathcal{S}$, let W solve

$$\begin{aligned} dW_t &= \gamma W_t dt - dA_t + \phi_c dB_t, & W_t &\leq \bar{W}, \\ \int_0^t \mathbf{1}_{\{W_s < \bar{W}\}} dA_s &= 0. \end{aligned}$$

The one-sided Skorokhod problem has a pathwise unique solution because the drift is Lipschitz and the volatility is constant and nondegenerate. The process A is the minimal upper-reflection process and is the pre-spinoff compensation process.

For every $T < \infty$, Itô's formula applied to $e^{-r(t \wedge \tau_*)} W_{t \wedge \tau_*}$ gives

$$\mathbb{E} \left[\int_0^{T \wedge \tau_*} e^{-rs} dA_s \right] \leq w_0 + \frac{\gamma - r}{r} \bar{W}.$$

Hence the reflected payout process has finite expected discounted variation. For $\tau_\varepsilon := \inf\{t : W_t \leq \varepsilon\}$, F' is bounded on $[\varepsilon, \bar{W}]$, so the martingale in (E.82), stopped at $T \wedge \tau_* \wedge \tau_\varepsilon$, is square-integrable. Letting first $T \uparrow \infty$ and then $\varepsilon \downarrow 0$ is valid because W is bounded before τ_* , $F(\varepsilon) \rightarrow F(0) = 0$, and the post-spinoff stand-alone contract keeps promised utility in $[0, \bar{W}_i]$. Since P_i is bounded on this interval, its expected discounted terminal value vanishes as $T \rightarrow \infty$. Every finite-variation inequality in (E.82) now binds: the continuation ODE holds before stopping, continuous payouts occur only where $F' = -1$, the initial payout lies in the linear tail, and the terminal payoff equals the binding obstacle. Therefore the localized identity passes to the limit with equality and gives payoff $F(W_0)$. An optimal contract exists, and the unique value and all the properties stated in Theorem 2 follow.

E.3 The Baseline Conglomerate without Spinoff

As in Appendix E.1, we have the following theorem for the conglomerate without spinoff.

Theorem 3. *Set $\mu_c = \mu_a + \mu_b$ and let ϕ_c be given by (B.33). The baseline conglomerate value P_c is the unique concave viscosity solution of*

$$\max \left\{ \frac{\phi_c^2}{2} P_c'' + \gamma W P_c' - r P_c + \mu_a + \mu_b, -1 - P_c' \right\} = 0, \quad (\text{E.84})$$

subject to $P_c(0) = 0$ and $\lim_{W \rightarrow \infty} P_c(W)/W = -1$. It is globally C^2 , strictly concave below a unique payout boundary $\bar{W}^c$, is linear above that boundary, and satisfies

$$P_c'(\bar{W}^c) = -1, \quad P_c''(\bar{W}^c) = 0, \quad P_c(\bar{W}^c) = \frac{\mu_a + \mu_b - \gamma \bar{W}^c}{r}. \quad (\text{E.85})$$

The optimal sensitivity pair is (β_a^, β_b^*) .*