Profitability of Frequency Reward Programs

Xing Li and V. Brian Viard*

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Abstract

We study the profitability of frequency reward programs under very general conditions. We begin by examining the profitability of a monopolist offering a generalized buy-one-get-one-free reward program with price $p + \delta$ to customers not holding a credit and $p - \delta$ to customers holding one. Spot pricing ($\delta = 0$) dominates any form of credit-specific pricing ($0 < \delta \leq p$) including a classic buy-one-get-one-free program ($\delta = p$). This result is robust to a multiple-credit-multiple-reward program, flexible cost structures, and most importantly, fully-flexible specifications of customer heterogeneity. We then identify specific conditions under which a frequency reward program can outperform spot pricing and provide three examples: new products or markets, third-party payment, and finite-credit expiration. We provide intuition and managerial implications of these results and reconcile previous findings of profit-enhancing reward programs.

Keywords: customer reward programs, finite expiration, transition path, pricing, dynamic competition

*Li (E-mail: xingli@gsm.pku.edu.cn): Guanghua School of Management, Peking University, Viard (E-mail: brian-viard@ckgsb.edu.cn): Cheung Kong Graduate School of Business. We thank Hongbin Cai, Yuxin Chen, Anthony Dukes, Yuetao Gao, Kinshuk Jerath, Mengze Shi, Baohong Sun, Miguel Villas-Boas, Yi Xiang, Xing Zhang, and seminar participants at Peking University, Xiamen University, Fudan University, Hong Kong Quant Marketing Online Series, and Summer Institute in Competitive Strategy for helpful feedback and suggestions. All errors are ours.

1 Introduction

Frequency reward programs are ubiquitous across a broad array of industries, including airlines, hotels, groceries, retailers, and coffee shops. The defining feature of a frequency reward program is that it rewards past purchases by basing prices on cumulative historical purchases; a typical structure is a buy-X-get-one-free (BXG1) program. Although these programs have been widely studied in marketing and economics, whether they are a profitable marketing tool depends on specific contexts such as behavioral preferences, particular competitive landscapes, and finite-credit expiration (Kim et al., 2001; Lu and Moorthy, 2007; Rossi, 2018; Sun and Zhang, 2019). Most other studies examine the design properties of reward programs conditional on their adoption (see Bombaij and Dekimpe (2020) for a survey). What remains understudied, as far as we know, is the profitability of a frequency reward program in a simple yet general setting with forward-looking, infinitely-lived customers. An infinite-horizon framework fully accommodates the dynamic decisions of customers facing a reward program. This serves as a benchmark for understanding the profitability of frequency reward programs under general conditions and, as a point of comparison, helps to isolate why they are profitable in specific contexts.

In this paper, we analytically examine a firm's incentive to offer a reward program in an infinite-horizon setting with very general customer heterogeneity and endogenous purchase frequency. Specifically, we ask whether a firm can realize higher profits by offering a reward program than by charging standard, period-by-period spot prices? We begin by examining a monopolist with zero marginal cost offering a generalized buy-one-get-one-free program: price $p + \delta$ to customers without a credit and $p - \delta$ to those with a credit.[1] We consider the steady state of an infinite-horizon model with forward-looking customers.[2] The key finding is that spot pricing ($\delta = 0$) dominates all forms of credit-specific pricing ($0 < \delta \leq p$), including a classic buy-one-get-one-free (B1G1) program ($\delta = p$), under a very weak distributional assumption of customer preferences, that the survival function (demand) is "not too convex". This assumption is satisfied by most continuous distributions, including those commonly used in empirical work.

Why does the generalized reward program realize lower profits than spot pricing under the "not too convex" assumption? The reward program endogenously creates two customer groups: those holding a credit and those not. In a mature market (steady state), the amounts purchased from the two groups are equal – the number of customers redeeming a credit equals the number earning one. Since the prices charged to these two groups ($p + \delta$ and $p - \delta$) average to the spot price p, the relative profitability of the reward

[1] This accommodates any prices p_0 and p_1 to those without and with a credit respectively: set $p = \frac{p_0+p_1}{2}$ and $\delta = \frac{p_1-p_0}{2}$.

[2] Although customers' time horizons are infinite, the model is equivalent to customers having finite, but uncertain, shopping lifespans (Blanchard, 1985)

program depends only on the total volume purchased relative to that under spot pricing, or equivalently, the purchase probability. We demonstrate that the aggregate purchase probability under the reward program equals the harmonic mean of the two groups' purchase probabilities. Under the "not too convex" assumption of customers' valuations, this harmonic mean is below the purchase probability under spot pricing. In other words, the higher price ($p + \delta$) faced by those without a credit dampens demand by more than the lower price ($p - \delta$) faced by those with a credit increases it.

Our model implicitly accommodates credit-specific pricing, where rewards may be discounted rather than free. The intuition, and therefore the results, also extend to frequency reward programs with other commonly observed features, including a non-zero marginal cost, a multiple-credit-multiple-reward program (a buy-X-get-Y-free, or BXGY, program). Most importantly, the result extends to an arbitrarily flexible pattern of customer heterogeneity, thereby ruling out price discrimination between frequent and infrequent customers as a motivation for implementing reward programs. Any of these features can be combined, and the reward program's underperformance carries through. This accommodates, for example, a tiered reward program that combines a BXGY program with credit-specific pricing for different units of Y. We also show that our main result is robust to "present bias" in the form of quasi-hyperbolic discounting.

We then consider exceptions in which a frequency reward program can yield higher profits than spot pricing. To understand how these scenarios arise, it is helpful to summarize the three reasons that drive the reward program's underperformance in the baseline model. First, since customers are in a steady state, the number of customers holding and not holding a credit are equal, and the firm faces no intertemporal tradeoff between short-term and long-term profitability. Second, the purchase amount from customers earning a credit (paying more) and those redeeming a credit (paying less) are balanced. Third, the "not too convex" property ensures that the aggregate purchase probability under spot pricing exceeds that in the reward program. Based on these, we demonstrate three circumstances, each corresponding to a violation of one of these pillars, under which a reward program may yield higher profits than spot pricing.

The first is a new product introduction or the introduction of a reward program into a pre-existing market. In this case, the steady-state condition is violated because all customers initially have zero credits. During the transition path towards the steady state, more purchases come from customers earning a credit (and paying the price premium) than from those redeeming a credit (and receiving the discount). The reward program allows the firm to exploit this imbalance in favor of the higher price, while spot pricing treats them equally. Once a market is mature and a steady state is reached, this advantage disappears.

The second is third-party payment, such as in a frequent-flier program, where a customer earns credits on a business trip paid by an employer. In this case, even if the market is in a steady state, the purchase amount at the higher price exceeds that at the lower price. Customers accumulate points during employer-funded travel, with the employer paying the premium price regardless of credit balance, and strategically reserve redemption for personal use. The reward program allows the firm to price discriminate across purchase occasions: charging a higher price on more inelastic, employer-funded purchases (as well as on occasions when customers earn a credit on a personal trip and are willing to pay more) and a lower price for price-sensitive personal trips (when customers redeem a credit). Spot pricing does not allow such price discrimination.

The third is finite-credit expiration. In this case, even if the market is in a steady state and the purchase volumes from customers with and without a credit are equal, the aggregate purchase probability in the reward program exceeds that under spot pricing. Finite expiration achieves this by effectively "bundling" transactions. To illustrate, compare a B1G1 program in which credits expire in one period to spot pricing. Spot pricing involves period-by-period purchases, and the firm sells only to customers with current, realized valuations above the price. The finite-expiration B1G1 program, in contrast, bundles a current spot purchase with a future (next-period) consumption. Since valuations for future consumption are uncertain at the time of purchase, customers evaluate them based on their expected rather than realized valuations. Since expected valuations are more similar across customers than are realized, bundling the purchases alleviates the firm's price-sales tradeoff, allowing it to charge a relatively higher price without losing commensurate sales. This advantage is absent when credits do not expire, as customers optimally delay redemption until they realize a sufficiently high valuation, effectively "unbundling" the transactions.

The remainder of the paper is organized as follows. The next section reviews related studies. Section 3 presents the main results, and Section 4 presents the exceptions when the reward program outperforms spot pricing. Section 5 concludes.

2 Related Literature

Our work relates to three branches of literature on reward program profitability: theoretical rational models, theoretical behavioral models, and empirical analyses. Here, we review each branch and identify our contribution.

2.1 Theoretical rational models

Several analytical papers examine the design features of reward programs conditional on their being offered. These include reward timing (immediate- versus delayed-redemption), reward type (in-kind products, unrelated products, and price discounts), and rules for accumulating and redeeming points (Kopalle et al., 2012; Chun et al., 2020a; Kim et al., 2021). These decisions depend on customer attributes, including preference heterogeneity across individuals and variation in preferences over time (Shin and Sudhir, 2010), rate of time preference, and frequency of shopping incidents (Sun and Zhang, 2019). Two recent papers examine the incentive to offer a reward program in a monopoly setting; both of which identify finite-credit expiration as a condition under which a frequency reward program can enhance profits.

Sun and Zhang (2019) studies an infinite-horizon model in which a monopolist can choose whether to offer a reward program with finite-credit expiration. Infinitely-lived customers are heterogeneous along two binary dimensions: product valuation and exogenous market-participation frequency. A reward program is more profitable than spot pricing when purchase frequency and valuations are negatively correlated. The negative correlation makes customers more homogeneous under the reward program than under static pricing because high-valuation customers purchase less often than low-valuation customers and are therefore more likely to lose an expiring credit. The increased uniformity relaxes the price-sales tradeoff for the firm. We also identify finite-credit expiration as a condition under which reward programs can outperform static pricing, but for a different reason. Finite-credit expiration expands demand for future products as customers make decisions based on expected rather than realized value. Relative to this paper, we show that the reward program can outperform spot pricing without customer heterogeneity. Relatedly, we show the necessity of finite-credit expiration by proving the sub-optimality of offering a reward program without it.

Liu et al. (2021) also finds that finite-credit expiration is one condition under which a BXG1 program can outperform static pricing. Using an infinite-horizon model with customers whose valuations are Bernoulli-distributed, the paper shows that a frequency reward program without finite-credit expiration cannot improve firms' profit, but that one with finite-credit expiration can.[3] We complement this paper by relaxing the distributional assumptions, generalizing the results, and providing an alternative argument, the smoothing of valuations, to understand the generalization. We also provide specific conditions required for a reward program to outperform spot pricing and provide examples, in addition to finite-credit expiration, to illustrate.

[3]Some of the paper's results rely on numerical simulations. Liu et al. (2021, p. 1840) states (using the acronym "BXGO" for a buy-X-get-one-free program) that, in the absence of these conditions: "We also conducted extensive numerical studies and did not find any parameter set under which the BXGO program improves the firm's profit." We provide a corresponding analytical proof in Section 3.4.

The logic of our finite-expiration results is similar to the smoothing of valuations across different customer segments in the classical bundling literature (Adams and Yellen, 1976; Bakos and Brynjolfsson, 1999, 2000; Stremersch and Tellis, 2002) and models of advance purchasing (Shugan and Xie, 2000; Xie and Shugan, 2001), in which a monopolist increases profits by pre-selling units, as the future expected value is more uniform than the later spot values. This allows the monopolist to sell to more customers, and this more than compensates for the fact that the advance purchase price lies below the spot price. Alexandrov and Özlem Bedre-Defolie (2014) makes a similar analogy between advance purchases and the traditional bundling literature, as we do for finite-expiration reward programs. In contrast to our model, these papers consider two-period models in which customers can only purchase at the spot price or the previously-paid advance price in the second period.[4] We show that this intuition carries over to a fully dynamic model in which customers can qualify for a reward in any period but face finite-credit expiration.

Besides the new insights on finite-expiration rewards, our model allows for more general customer heterogeneity and endogenous purchase frequencies, provides analytical results that guarantee robustness, and identifies specific conditions under which reward programs can outperform spot pricing. We provide intuition for these results and examine their generalizability along other reward-program dimensions, such as BXGY programs, credit-specific pricing, and time-inconsistent preferences.

There are a few papers related to the other exceptions we identify in which reward programs can outperform spot pricing (new markets and third-party payments). In the context of a new market, when firms launch a reward program, they engage in "revenue management" by realizing profits earlier, and the earned credits are liabilities to customers (Chun et al., 2020b). Concerning third-party payments, Shugan (2005) argues that reward programs, especially in travel, may take advantage of a principal-agent problem between decision makers and payers, although Basso et al. (2009) argues that this may not hold in duopoly competition. We identify the fundamental properties that must be present for these exceptions to succeed.

Other papers consider rationales for reward programs that we do not examine. Shugan (2005) argues that price discrimination on redemption efforts, shifting revenues and costs, and implementing quantity discounts are motivations for reward programs. In a competitive environment, launching a reward program increases customers' stickiness and softens price competition by affecting customers' search process (Kuksov and Zia, 2021; Ke et al., 2026), inducing switching costs (Kim et al., 2001; Singh et al., 2008; Bazargan et al., 2018, 2020), and managing limited capacity (Kim et al., 2004). Most of these arguments are formalized in a two-period model that implicitly assumes finite-credit expiration. It remains uncertain whether these insights

[4] The finite-period setting is necessary for advance purchases to increase profits. As in our model, if customers could wait to purchase until their valuation is high, the averaging of expected future values would unravel.

apply in the case of an infinite-horizon framework.

2.2 Theoretical behavioral models

Another branch of literature argues that reward programs exploit customers' non-rational behavior. These include reward effort to alleviate guilt (Kivetz and Simonson, 2002), reward effort as a perceived advantage over others (Kivetz and Simonson, 2003), unused credits in the form of "slippage" (Lu and Moorthy, 2007), different mental accounting for credits and cash (Zhang and Breugelmans, 2012; Stourm et al., 2015), increased purchases due to "points pressure" (Taylor and Neslin, 2005; Kivetz et al., 2006; Kopalle et al., 2012; Wang et al., 2016), "medium maximization" (Hsee et al., 2003), "rewarded-behavior" mechanisms (Taylor and Neslin, 2005; Drèze and Nunes, 2011; Kopalle et al., 2012), extra utility from point redemption (Rossi, 2018; Liu et al., 2021), the existence of hurdle costs for point redemption (Liu et al., 2021), and other bounded-rationality effects (Liu and Ansari, 2020). We rule out behavioral elements to establish a benchmark of reward program profitability. Any behavioral effects must be large enough to compensate for the inferiority of the reward program under rationality.

2.3 Empirical analyses

Empirical studies find mixed results on reward program effects. Sharp and Sharp (1997) finds no evidence that a retailer reward program increases repeat purchases, while Iyengar et al. (2022) finds a reward program increases purchases but not necessarily profits. Liu (2007) finds that a convenience-store reward program does not change the purchase behavior of ex-ante frequent buyers but does accelerate purchase frequency and size for less-frequent buyers. Lal and Bell (2003) finds that a grocery reward program is profitable because it reduces "cherry-picking" by infrequent customers enough to compensate for rewards paid out to frequent customers. Heerde and Bijmolt (2005) decomposes the effect of different promotions on reward program members versus non-members and assesses their profitability. Gopalakrishnan et al. (2021) finds that the introduction of a hair salon reward program increases profits by reducing attrition.

For reasons why reward programs might outperform spot pricing, Orhun et al. (2022) provides empirical evidence for the moral-hazard argument in the context of third-party payment. Sun and Zhang (2019) examines how the distribution of purchase frequencies, prices, and reward size determine the optimal expiration term for credits. In the context of competition, Rossi and Chintagunta (2023) finds price increases in the later periods after a gas station adopts a reward program, consistent with "lock-in" due to switching costs. A number of empirical papers examine switching costs conditional on a reward program being offered (Lewis, 2004; Hartmann and Viard, 2008; Orhun et al., 2022).

Finally, our paper relates to structural empirical models of switching costs and reward programs, (Lewis, 2004; Hartmann, 2006; Hartmann and Viard, 2008; Dubé et al., 2009; Kopalle et al., 2012) as our distributional assumptions accommodate these commonly-used models.

3 Main Results

We begin by considering homogeneous customers with independent random utility across periods and describe customers' decisions under a "generalized B1G1" reward program. In a typical reward program, customers pay a price p if they have zero credits, and a price 0 when they have one credit. We allow the price with one credit to be non-zero: customers pay a price $p_0 = p + \delta$ if they have zero credits and $p_1 = p - \delta$ if they have one credit. This generalized B1G1 program nests the classic B1G1 reward program ($\delta = p$), spot pricing ($\delta = 0$), and credit-specific pricing ($0 < \delta < p$).

Our main result is that, given a weak assumption about the random-utility distribution, even though the firm has two choice variables (p and δ), it will always choose $\delta = 0$ (spot pricing). Firms cannot do better by dividing customers into two groups based on their credit balances and charging them different prices. We describe the intuition that underlies this result. We then show that the underperformance of reward programs extends to much more general specifications of the reward program structure and demand. These include a nonzero marginal cost, a higher-threshold-multiple-reward program (a buy-X-get-Y free or BXGY program), time-inconsistent customer preferences, and very general specifications of customer heterogeneity. We conclude this section by eliminating the distributional assumption and showing that our main result holds with no restriction on the preference distribution for the baseline case and most extensions.

3.1 Model setup

We consider a monopolist facing a group of (ex-ante) homogeneous customers. In each period, each customer realizes a random utility $v \sim F(\cdot)$, independent and identically distributed across customers and over time. Let $G(v) = 1 - F(v)$ be the survival function of the distribution. Customers are infinitely lived and maximize their lifetime benefit by applying the discount factor β. This does not mean that customers literally live forever. Geometric discounting of utility is consistent with a finite but uncertain lifetime or "shopping lifespan" (Blanchard, 1985).[5] We relegate all proofs to Appendix A unless otherwise specified.

[5]Suppose that a customer has an expected lifespan of $\frac{1}{\rho}$ (probability ρ of death in each period with $0 < \rho < 1$). The customer's discount factor can be redefined as $\beta = \lambda(1 - \rho)$ where λ is the time discount factor conditional on surviving to the

We consider a generalized B1G1 reward program in which the customer's price depends on whether they hold a credit ($s \in \{0, 1\}$). They pay a higher price ($p_0 = p + \delta$) to consume if $s = 0$ and accumulate a credit, but pay a lower price ($p_1 = p - \delta$) to consume if $s = 1$ and redeem the credit. The customer's decision rule can be described by a Bellman equation. Let $u_a(v, s)$ be the payoff for a customer with realized random utility v holding credits s and who takes the action $a \in \{0, 1\}$ representing no-purchase and purchase:

$$u_1(v, s) = v - p_s + \beta \cdot w(1 - s) \tag{1}$$

$$u_0(v, s) = \beta \cdot w(s), \tag{2}$$

where $w(s)$ is the continuation value in the next period as a function of the state variable:

$$w(0) = E \max (u_1(v, 0), u_0(v, 0)) \tag{3}$$

$$w(1) = E \max (u_1(v, 1), u_0(v, 1)) . \tag{4}$$

For any values of $\{G(\cdot), \beta, p\}$, the value functions can be solved using the above two sets of equations.

Given the value functions, and therefore utility, customers' consumption probabilities under the two states ($q_s, s \in \{0, 1\}$) are:

$$q_0 = \Pr (u_1(v, 0) > u_0(v, 0)) = G(p + \delta - \beta\Delta w) \tag{5}$$

$$q_1 = \Pr (u_1(v, 1) > u_0(v, 1)) = G(p - \delta + \beta\Delta w), \tag{6}$$

where $\Delta w = w_1 - w_0$. To solve for the steady-state probability of the two states, let (r_0, r_1) be the probability of customers holding zero and one credit, respectively, with $r_0 + r_1 = 1$. In a steady-state:

$$r_0 = q_1 r_1 + (1 - q_0) r_0, \tag{7}$$

which gives:

$$r_0 = \frac{q_1}{q_1 + q_0}. \tag{8}$$

$$r_1 = \frac{q_0}{q_1 + q_0}. \tag{9}$$

next period. ρ need not necessarily refer to physical death but rather to a "shopping lifespan". For example, a customer may face some probability that they will no longer need the firm's product because they move or their circumstances change.

Thus, the profit from the reward program is:

$$\pi(p, \delta) = (p + \delta) \cdot (r_0 q_0) + (p - \delta) \cdot (r_1 q_1), \tag{10}$$

where the first term represents the profit from those with $s = 0$ (type-0 customers), and the second term from those with $s = 1$ (type-1 customers). Given that in the steady state specified in (8) and (9):

$$r_0 q_0 = r_1 q_1 = \frac{q_0 q_1}{q_0 + q_1}, \tag{11}$$

the profit function in Equation (10) can be simplified to:

$$\pi(p, \delta) = p \cdot (r_0 q_0 + r_1 q_1) = p \cdot \frac{2 q_0 q_1}{q_0 + q_1} = p \cdot Q(p, \delta). \tag{12}$$

The identity $r_0 q_0 = r_1 q_1$ is a direct consequence of the steady-state, or mature-market, assumption: the purchase amount from type-0 customers must equal the purchase amount from type-1 customers. In a mature market, the number of customers who switch states (from type-0 to type-1 and vice versa) is equal, which makes the state distribution stable.

Spot pricing is a special case of the generalized B1G1 program with $\delta = 0$. In this case, $p_0 = p_1 = p$ and customers face an identical environment regardless of credit value. Consequently, $u_a(v, 0) = u_a(v, 1)$, $w(0) = w(1)$, and $q_0 = q_1 = G(p)$. Thus:

$$Q_S(p) = Q(p, 0) = G(p) \tag{13}$$

$$\pi_S(p) = \pi(p, 0) = p \cdot G(p). \tag{14}$$

3.2 Comparing reward program and spot pricing

Three observations about the profit structure of the reward program facilitate an analytical comparison to spot pricing without explicit solutions. First, there are two endogenously-formed groups under the reward program: those with $s = 0$ and those with $s = 1$. In the steady state, purchase amounts in these two groups are the same. Second, one group is charged a price $p + \delta$ and the other $p - \delta$, which, given the equal sizes of the groups, offset each other in the profit function. These two observations imply that comparing profits under the reward program and under spot pricing collapses to a comparison of the average purchase probability under the reward program, $Q(p, \delta)$ in Equation (12), and the purchase probability under spot pricing, $G(p)$ in Equation (14).

The third observation is that the average purchase probability under the reward program simplifies dramatically, facilitating an easy comparison to the purchase probability under spot pricing. Under the reward program, the average purchase probability is the harmonic mean of the purchase probabilities in the two states, q_0 and q_1, as seen in Equation (12). The average of their arguments ($p+\delta-\beta\Delta w$ and $p-\delta+\beta\Delta w$ shown in Equations (5) and (6)) is p, which conveniently does not depend on the complicated parts of the value functions or on δ. The purchase probability under spot pricing also depends only on the price p. This makes it possible to impose a weak convexity condition and apply Jensen's inequality to compare the two realized demands, and therefore profits, directly:

Assumption 1. *The reciprocal of the survival function, $H(v)$, is convex, where:*

$$H(v) = \begin{cases} 1/G(v) & \text{if } G(v) > 0 \\ +\infty & \text{if } G(v) = 0. \end{cases} \tag{15}$$

This assumption is met by most commonly-used continuous distributions (as shown in Appendix B), including those used in empirical structural estimation (i.e., type-I extreme value distribution). It is implied by, and therefore weaker than, other commonly-used assumptions, such as an increasing hazard rate (Lei et al., 2025), log-concavity of the survival function (Bar-Isaac and Shelegia, 2023), log-concavity of the probability density function (Chen et al., 2019; Gong and Huang, 2025), and concave survival function (Jain and Li, 2018).[6] Given this assumption, our main result is:

Proposition 1. *Under Assumption 1, for any value of p:*

$$Q(p,\delta) \leq Q(p,0), \tag{16}$$

and correspondingly:

$$\pi(p,\delta) \leq \pi(p,0) = \pi_S(p), \tag{17}$$

where $Q(p,\delta)$ and $\pi(p,\delta)$ are defined in Equation (12).

[6] The only paper that mentions the convexity of the reciprocal of demand, as far as we know, is Caplin and Nalebuff (1991), which shows that it implies quasi-concavity of the profit function and a well-defined profit-maximization problem for the firm.

Proof. Since $H(v)$ is convex, Jensen's inequality implies:

$$H(p) \leq \frac{1}{2}\left(H(p+\delta-\beta\Delta w)+H(p-\delta+\beta\Delta w)\right). \tag{18}$$

Substitute to get:

$$\frac{1}{G(p)} \leq \frac{1}{2}\cdot\left(\frac{1}{G(p+\delta-\beta\Delta w)}+\frac{1}{G(p-\delta+\beta\Delta w)}\right)=\frac{1}{2}\cdot\left(\frac{1}{q_0}+\frac{1}{q_1}\right). \tag{19}$$

Taking the reciprocal of both sides yields Equation (16). □

To understand Assumption 1, it is useful to consider a stronger version: that demand, $G(v)$, is concave. The reward program endogenously creates two groups of demand: $G(p+\delta-\beta\Delta w)$ and $G(p-\delta+\beta\Delta w)$. Concave demand implies that the arithmetic average of these two demands is below the demand evaluated at the average of their arguments, which equals the spot price. Assumption 1 is a weaker condition – that demand is not "too convex". As the profit comparison is based on a harmonic rather than an arithmetic average, the weaker assumption is sufficient for the underperformance of the reward program. Figure 1 demonstrates how $G(\cdot)$ can be fairly convex and demand under the reward program is still lower than demand under spot pricing. The thick, solid curve shows demand, $G(\cdot)$, evaluated between the price to type-1 and type-0 customers: $(p-\delta+\beta\Delta w)$ and $(p+\delta-\beta\Delta w)$. The height of the short-dashed, left and right vertical bars corresponds to the demand from the two types, q_0 and q_1, respectively. The height of the solid vertical line intersecting the two dashed lines is the harmonic mean of q_0 and q_1.[7] The height of the vertical, dash-dotted vertical line (located at p) is demand under spot pricing, $G(p)$, which lies above the harmonic mean. As shown in the figure, as long as the demand curve is "not too convex", Assumption 1 is met and the reward program is less profitable than spot pricing.

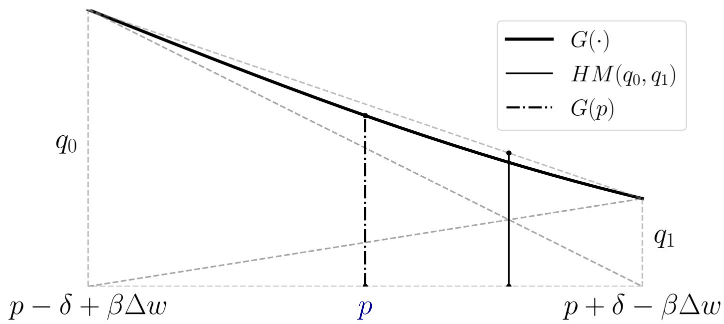
A direct implication of this result is that the monopolist's profits are lower under a classic B1G1 reward program ($\delta = p$) than under spot pricing. Specifically, let π_R^* and π_S^* be the optimal profits under the reward program and spot pricing, respectively, then:

Corollary 1. *Under Assumption 1, applying the result in Equation (12):*

$$\pi_R^* = \max_p \pi(p,p) \leq \max_p \pi(p,0) = \pi_S^*. \tag{20}$$

[7] The geometric representation of the harmonic mean of two parallel edges in a trapezoid can be proved using similarity of triangles (see https://en.wikipedia.org/wiki/Harmonic_mean).

Figure 1: Illustration of “not too convex” for normal distribution



Notes: The thick, solid curve is the survival function of a standard normal distribution evaluated between the prices to type-0 customers ($p - \delta + \beta\Delta w$) and type-1 customers ($p + \delta - \beta\Delta w$). The general property that the vertical, solid line (harmonic mean of q_0 and q_1), located at the intersection of the dashed lines, is shorter than the vertical, dash-dotted line (survival function evaluated at the mid-point of the two prices or p), holds globally for most continuous distributions.

This corollary follows directly from the fact that profits are greater under spot pricing for *any* price; therefore, profits must be greater under the optimal price. A firm charging price $p_0 = 2p$ and $p_1 = 0$ under a B1G1 reward program can always earn higher profits by charging $p_0 = p_1 = p$ under spot pricing.

In summary, under very general conditions, the firm cannot do better by artificially dividing the homogeneous customers into two groups that are less similar and charging them different prices. This is because customers respond endogenously to prices. They switch states optimally over time to avoid the firm's rent extraction. Although under price discrimination firms can generally do better with additional pricing dimensions, that is not the case here because the types, based on the number of credits they hold, are endogenously determined. There is a single customer type (distribution).

3.3 Extensions

Our main result, Proposition 1, can be extended to a flexible specification of customer heterogeneity. It also extends along several other dimensions, including a non-zero marginal cost, a higher-threshold-multiple-reward (BXGY) program, and a form of time-inconsistent customer preferences. Any of these extensions can be combined with any of the others. For example, a reward program with tiers would combine a BXGY program with credit-specific pricing, customer heterogeneity, and potentially non-zero marginal cost. In fact, all these elements can be present in a single model, and the results hold.

3.3.1 Heterogeneity

We extend the model to a rich specification of customer heterogeneity in which customers' types are captured by a stochastic dimension, v, and show that our main result still holds. Consider a finite mixture of the type distribution in which there are K types of customers parameterized by $\{(G^k(\cdot), \beta^k)\}_{k=1}^K$, each with probability λ^k and $\sum_k \lambda^k = 1$. The profit is:

$$\pi(p, \delta; \lambda) = \sum_k \lambda^k \cdot \pi^k(p, \delta), \tag{21}$$

where $\pi^k(p, \delta)$ is the profits from type-k customers:

$$\pi^k(p, \delta) = (p + \delta) \cdot (r_0^k q_0^k) + (p - \delta) \cdot (r_1^k q_1^k), \tag{22}$$

and $\{r_s^k, q_s^k\}$ are state distribution and purchase probabilities for type-k customers. Applying Proposition 1 separately to each type, the result holds as:

$$\pi^k(p, \delta) \leq \pi^k(p, 0). \tag{23}$$

Therefore, it holds in aggregate:

$$\pi(p, \delta; \lambda) \leq \pi(p, 0; \lambda). \tag{24}$$

Since this holds at any price, it holds at the optimal price.

A proposed benefit of a reward program is price discrimination, as it can charge different effective prices to customers with different purchase frequencies: frequent customers pay a lower price than infrequent ones. However, this is not effective because purchase frequencies are endogenously determined. Instead of charging different prices based on the exogenous types k, the firm can only charge different prices based on the credits customers hold, which are endogenously formed. Price discrimination based on these endogenous types is ineffective.

Sun and Zhang (2019) shows, with Bernoulli-distributed random utility, that a reward program achieves higher profits than spot pricing if: 1) credits have a finite expiration, and 2) preference heterogeneity is such that there is a negative correlation between valuation and shopping probability. Our finding complements this by showing that finite expiration is necessary to obtain higher profits (we elaborate on this in Section 4.3). In particular, our results show that a negative correlation between valuation and purchase probability (captured by the joint distribution of $G^k(\cdot)$ and β^k) does not guarantee that the reward program achieves higher profits.

3.3.2 Credit-specific pricing

The generalized reward program with two prices generalizes the classic reward program to credit-specific pricing ($0 < \delta < p$). Therefore, our main result is robust to any credit-specific pricing, either for earning a credit or redeeming a reward.

3.3.3 Non-zero marginal cost

Assume that the monopolist incurs a constant marginal cost, c, to serve a unit of product. This occurs at the time of consumption rather than purchase (e.g., in a coffee reward program, the cost is incurred when the coffee is served). Customers face the same problem as in the base model, but firms incur this marginal cost. The profit function becomes:

$$\pi(p, \delta; c) = (p + \delta - c)(r_0 q_0) + (p - \delta - c)(r_1 q_1) \tag{25}$$

$$= (p - c) \cdot (r_0 q_0 + r_1 q_1) = (p - c) \cdot Q(p, \delta). \tag{26}$$

The equality follows from the fact that $r_0 q_0 = r_1 q_1$ in the customer's problem (Equation (11)). As $Q(p, \delta) \leq Q(p, 0)$ from Proposition 1, multiplying both sides by $(p - c)$, we have $\pi(p, \delta; c) \leq \pi(p, 0; c)$. Since this holds at all prices, optimal profits under the reward program are less than those under spot pricing.

3.3.4 Higher-threshold-multiple-reward (BYGZ)

A natural extension of the main result is a frequency reward program with a qualifying threshold higher than one, and the potential to earn more than one reward. In this case, customers can have $s \in \{0, 1, ..., X - 1\}$ credits in balance, which cycle back to zero. For example, a buy-Y-get-Z-free program would have $X = Y + Z$ states, with $p_s = p$ for $s < Y$ and $p_s = 0$ for $Y \leq s < Y + Z$. We consider a generalized reward program, with the firm charging $\{p_0, p_1, ..., p_X\}$ based on a customer's credit status. Customers' payoff functions are:

$$u_1(v, s) = v - p_s + \beta \cdot w\left(g(s)\right) \tag{27}$$

$$u_0(v, s) = \beta \cdot w(s), \tag{28}$$

with state transition function $g(s) = s + 1$ if $s < X$, and $g(s) = 0$ if $s = X$. The value function is:

$$w(s) = E \max\left(u_1(v, s), u_0(v, s)\right). \tag{29}$$

The choice probabilities (q_s) and steady-state distribution (r_s) are:

$$q_s = \Pr\left(v > p_s - \beta \Delta w(s)\right) = G\left(p_s - \beta \Delta w(s)\right) \tag{30}$$

$$r_{(g(s))} = r_s \cdot q_s + r_{g(s)} \cdot (1 - q_{g(s)}), \tag{31}$$

where $\Delta w(s) = w\left(g(s)\right) - w(s)$, and $\sum_{s=0}^{X-1} \Delta w(s) = 0$. The profit is thus:

$$\pi(p_0, p_1, ..., p_{X-1}) = \sum_{s=0}^{X-1} p_s \cdot (r_s q_s). \tag{32}$$

We have a result analogous to that of the generalized B1G1 program:

Proposition 2. *Under Assumption 1, the generalized BYGZ reward program charging different prices under different states yields lower profits than spot pricing with the same average per-unit price:*

$$\pi(p_0, p_1, ..., p_{X-1}) \leq \pi(\bar{p}, \bar{p}, ...\bar{p}), \tag{33}$$

where

$$\bar{p} = \frac{1}{X} \sum_{s=0}^{X-1} p_s. \tag{34}$$

3.3.5 Quasi-hyperbolic discounting

Time-inconsistent preferences and lack of self-control are sometimes used to explain customers' intertemporal behaviors (Laibson, 1997; O'Donoghue and Rabin, 1999; Jain, 2012; Jain and Li, 2018; Amaldoss and Harutyunyan, 2023). We examine reward program profitability when customers are time-inconsistent and exhibit quasi-hyperbolic discounting. Consider naive customers who believe that their future behavior is determined by time-consistent preferences with discount factor β, whereas in the current period, their discount factor is smaller ($\tilde{\beta} < \beta$). Suppose the firm offers a generalized B1G1 program at prices $(p_0, p_1) = (p + \delta, p - \delta)$. The customer thinks his value function in the future, denoted as $w(s; \beta)$, is purely rational, determined by (1) –

(4). In the current period, however, their utility is:

$$\tilde{u}_1(v,s) = v - p_s + \tilde{\beta} \cdot w(1-s;\beta) \tag{35}$$

$$\tilde{u}_0(v,s) = \tilde{\beta} \cdot w(s;\beta). \tag{36}$$

Their purchase likelihood is:

$$\tilde{q}_0 = \Pr\left(\tilde{u}_1(v,0) > \tilde{u}_0(v,0)\right) = G(p+\delta-\tilde{\beta}\cdot\Delta w(\beta)) \tag{37}$$

$$\tilde{q}_1 = \Pr\left(\tilde{u}_1(v,1) > \tilde{u}_0(v,1)\right) = G(p-\delta+\tilde{\beta}\cdot\Delta w(\beta)), \tag{38}$$

and the steady-state distribution $(\tilde{r}_0, \tilde{r}_1)$ is determined as in Equations (8) and (9). The profit is thus

$$\pi(p,\delta;\tilde{\beta},\beta) = p_0 \cdot (\tilde{r}_0\tilde{q}_0) + p_1 \cdot (\tilde{r}_1\tilde{q}_1), \tag{39}$$

and we have a result analogous to Proposition 1:

$$\pi(p,\delta;\tilde{\beta},\beta) \le \pi(p,0;\tilde{\beta},\beta). \tag{40}$$

Since this holds at any price, it holds at the optimal price. Even if customers exhibit time-inconsistent preferences, it is not optimal for the firm to offer a generalized reward program.

3.4 Even weaker assumption

Although Assumption 1 applies to most continuous distributions, it does not apply to discrete ones (e.g., Bernoulli). This means that the main result (the underperformance of a reward program at any price) does not necessarily hold in these cases. However, the balance of purchase amounts in the steady state condition still holds, and it turns out that this property suffices to guarantee the underperformance of a reward program at the *optimal* price without any distributional assumptions:

Proposition 3. *Let $\pi_R(p_0, p_1)$ be the profit under a generalized reward program charging two prices in two states (i.e., $\pi_R(p_0,p_1) = \pi(\bar{p},\delta)$ in (12), where $\bar{p} = (p_0+p_1)/2$ and $\delta = (p_0-p_1)/2$), and π_S^* be the optimal profit under spot pricing:*

$$\pi_S^* = \max_p \pi_S(p) = \max_p p \cdot G(p), \tag{41}$$

we have:

$$\pi_R(p_0, p_1) \leq \pi_S^*. \tag{42}$$

Proof. Note that:

$$\pi_R(p_0, p_1) = p_0(r_0 q_0) + p_1(r_1 q_1) = r_0 p_0 G(p_0 - \beta\Delta w) + r_1 p_1 G(p_1 + \beta\Delta w). \tag{43}$$

The profit can be represented as a linear combination of two spot-pricing profits under different prices:

$$\pi_R(p_0, p_1) = r_0(p_0 - \beta\Delta w)G(p_0 - \beta\Delta w) + r_1(p_1 + \beta\Delta w)G(p_1 + \beta\Delta w) + \beta\Delta w\,(r_0 q_0 - r_1 q_1) \tag{44}$$

$$= r_0 \pi_S(p_0 - \beta\Delta w) + r_1 \pi_S(p_1 + \beta\Delta w), \tag{45}$$

where the last equality comes from the fact that in the steady-state, $r_0 q_0 = r_1 q_1$ (as in the baseline case). Finally, due to the optimality of π_S^*, both of these profits are below profits at the optimal price:

$$\pi_S(p_0 - \beta\Delta w) \leq \pi_S^* \tag{46}$$

$$\pi_S(p_1 + \beta\Delta w) \leq \pi_S^*, \tag{47}$$

so their linear combination is also:

$$\pi_R(p_0, p_1) = r_0 \pi_S(p_0 - \beta\Delta w) + r_1 \pi_S(p_1 + \beta\Delta w) \leq \pi_S^*. \tag{48}$$

□

This extends Proposition 3 of Liu et al. (2021), which proves the underperformance of a BXG1 program with "Bernoulli customers" given two constraints on parameter values. Our Proposition 3 relaxes both the distributional (Bernoulli) assumption and the parameter constraints. It is also robust to a nonzero marginal cost, a higher-threshold-multiple-reward program (a buy-Y-get-Z free program or BYGZ), and quasi-hyperbolic discounting. Because it holds only at optimal prices, it does not necessarily extend to a setting with heterogeneous types.

4 Exceptions

Three conditions underlie the reward program's sub-optimality relative to spot pricing: 1) the market is in a steady state with respect to the credits that customers hold, 2) the purchase amount from customers that hold a credit equals the amount from those that do not, and 3) the purchase probability in the reward program is below that under spot pricing. For a standard generalized reward program, the first condition is an assumption that implies the second condition, and the "not too convex" assumption implies the third condition. This section identifies three exceptions in which the reward program may outperform spot pricing, because one of these conditions does not hold.

The first is on a transition path before reaching a steady state, corresponding to a new product launch or introduction of a reward program into an existing market. The second is third-party payment, such as in airline frequent-flier programs, when business trips are paid by a customer's employer. The third is finite-credit expiration. For each of these, we demonstrate how they violate one of the conditions and how this may lead to higher profits under a reward program than under spot pricing. To illustrate, we show that a classic B1G1 program, when the firm charges a price of one (i.e., $p_0 = 1$ and $p_1 = 0$), yields higher profits than optimal spot pricing when customers' valuations are uniformly distributed on the unit interval ($v \sim U(0, 1)$).

4.1 Transition path

A reward program may outperform spot pricing along the transition path to a steady-state.[8] Consider the case in which the firm launches a new product, and all customers begin with zero credits ($r_0(1) = 1$ where $r_0(t)$ is the fraction of customers holding zero credits in period t). As time passes, $r_0(t)$ declines and the state distribution moves toward the steady-state level ($r_0(t) \to \bar{r}_0$). Along the transition path, there is more consumption from customers without a credit than from those with one. Before the steady state is reached, the firm can take advantage of the demand imbalance in favor of type-0 customers, who pay a higher price, and earn higher profits than it does in the steady state. Although once the steady state is reached, the reward program generates lower profit than spot pricing would, it can be more profitable during the transition path. If the firm has a sufficiently high discount factor, the reward program can be more profitable ex-ante (at $t = 1$).

To formalize this, consider a B1G1 program with $p_0 = 2p$ and $p_1 = 0$. Customers' purchase likelihoods

[8] We thank Miguel Villas-Boas for raising this point.

are the same as in the baseline analysis:

$$q_0 = G(2p - \beta\Delta w) \tag{49}$$

$$q_1 = G(\beta\Delta w), \tag{50}$$

where Δw is determined by the Bellman equations along the transition path. In each period t, the firm's profit from the reward program is:

$$\pi(p, r_0(t)) = 2p \cdot r_0(t) \cdot q_0. \tag{51}$$

The state transition is:

$$r_0(t+1) = r_0(t) \cdot (1 - q_0) + (1 - r_0(t)) \cdot q_1 \tag{52}$$

$$= q_1 + (1 - q_0 - q_1) \cdot r_0(t), \tag{53}$$

and $r_0(1) = 1$. To make the analysis easier, we consider the case with $q_0 + q_1 < 1$, which holds in most cases, and implies the following property:

Lemma 1. *Suppose $q_0 + q_1 < 1$, then $\{r_0(t)\}$ for $r_0(1) = 1$ is a decreasing sequence that converges to $\bar{r}_0 = \frac{q_1}{q_0+q_1}$.*

Given this, we can bound the profit along the transition path. Assume that the firm has a discount factor γ, then the discounted total profit at $t = 1$ is:

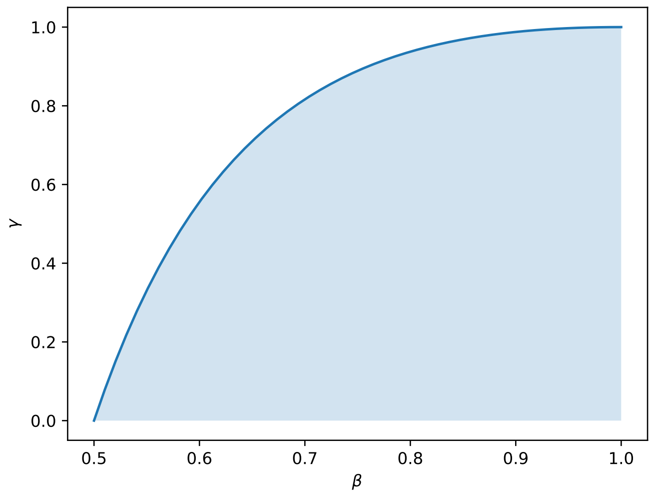
$$\Pi_R(p) = \sum_{s=1}^{+\infty} \gamma^{s-1} \cdot \pi(p, r_0(s)) \geq \pi(p, 1) + \frac{\gamma}{1-\gamma} \cdot \pi(p, \bar{r}_0), \tag{54}$$

where we bound the profits in all future periods by the profits in the steady state, as $r_0(s) \geq \bar{r}_0$. If the firm charges the optimal spot price, the total discounted profit is:

$$\Pi_S^* = \frac{\pi_S^*}{1-\gamma} = \pi_S^* + \frac{\gamma}{1-\gamma} \cdot \pi_S^*. \tag{55}$$

From the last section, we know that the firm's profit under a reward program is less than that under spot pricing in the steady state ($\pi(p, \bar{r}_0) \leq \pi_S^*$); however, the first-period profit ($\pi(p, 1)$) can be high enough that $\Pi_R(p) > \Pi_S^*$. To make this concrete, we provide an example of a B1G1 reward program with uniformly distributed valuations. For this example, the following proposition describes the conditions under which a

Figure 2: Comparison of B1G1 reward program and spot pricing profits along a transition path



Notes: Region of (β, γ) in which a reward program with valuation distribution $U(0,1)$, $r_0(1) = 1$, and $p_0 = 1$ earns higher profit than spot pricing.

reward program can generate higher profits than spot pricing:

Proposition 4. *For a B1G1 reward program with $v \sim U(0,1)$ and $r_0(1) = 1$, for any $\beta > \frac{1}{2}$ and $\gamma < \frac{1}{\beta}\left(2 - \frac{1}{\beta}\right)$, firms can earn more profits from a reward program at price $p_0 = 1$ than under spot pricing ($\Pi_R(0.5) \geq \pi_S^*$).*

Figure 2 displays the region of parameters under which the reward program can earn higher profits than spot pricing – the firm discount factor γ is small enough, and customers' discount factor β is large enough. A small γ means that firms value the immediate, higher flow profit more than the future, lower steady-state profits. A large β means that the difference in purchase probability between the reward program and spot pricing, and thus the steady-state profit, is smaller.[9]

4.2 Third-party payment

The second exception involves third-party payments in which the purchase amount from high-paying customers can exceed that from low-paying customers, even in a steady state. For example, in airline frequent flier programs, customers generally do not pay out of pocket for business travel. On these occasions, a cus-

[9] Note that $q_0 = G(2p - \beta\Delta w)$ and $q_1 = G(\beta\Delta w)$. A higher β means the two purchase probabilities are closer and their average is closer to the purchase probability under spot pricing.

tomer holding a credit ($s = 1$) pays the higher price, p_0, and keeps the credit for future personal consumption. In this environment, the reward program can outperform spot pricing. Because customers strategically forgo redemption on business trips, the volume of transactions from customers paying the higher price can exceed that from customers paying the lower price, even in a steady state. The reward program allows the firm to capitalize on this imbalance. By offering two price instruments, the firm can set the higher price, p_0, for employer-paid travel and leisure travelers earning a credit, while targeting the lower price, p_1, at leisure travelers redeeming a credit. The reward program facilitates price discrimination based on the exogenous distinction between business and leisure travel, whereas spot pricing imposes a uniform price across both.

To formalize, suppose that in each period, employer-paid consumption (a business trip) occurs with probability λ. On these occasions, the employer's willingness to pay is fixed at v_2. With probability $1 - \lambda$, the customer pays ("leisure" consumption), and their valuation from consumption is $v \sim F(\cdot)$. For simplicity, we assume that the domain of leisure valuation v is bounded from above by $\overline{v}$, and the employer's willingness to pay exceeds this bound:

$$v_2 > \overline{v}. \tag{56}$$

Under spot pricing with price $p < v_2$ (if $p > v_2$, profit is zero), the profit is:

$$\pi_S(p) = \lambda \cdot p + (1 - \lambda) \cdot p \cdot G(p), \tag{57}$$

where customers always buy on employer-paid occasions, but buy on leisure occasions only if $v > p$.

To solve for profits under the reward program, assume that customers obtain zero flow utility from employer-paid consumption, and flow utility $v - p_s$ from leisure consumption in state $s \in \{0, 1\}$. We focus on the classic reward program with $p_0 = 2p$ and $p_1 = 0$. The payoffs $(u_b, u_a(v, s))$ on employer-paid occasions ("b" for business) and leisure occasions with action $a \in \{0, 1\}$ are:[10]

$$u_b = \beta \cdot w(1) \tag{58}$$

$$u_1(v, s) = v - p_s + \beta \cdot w(1 - s) \tag{59}$$

$$u_0(v, s) = \beta \cdot w(s). \tag{60}$$

The value function is:

$$w(s) = \lambda \cdot u_b + (1 - \lambda) \cdot E \max\left(u_1(v, s), u_0(v, s)\right). \tag{61}$$

[10]To simplify, we assume that customers cannot accumulate more than one credit. Allowing the firm this option would make the reward program weakly even more favorable relative to spot pricing.

Although the value function includes employer-paid occasions that deviate from the benchmark case in the previous section, the payoff functions $u_a(v, s)$ are the same. The purchase probabilities under leisure consumption, q_0 and q_1, are also the same:

$$q_0 = \Pr\left(u_1(v, 0) > u_0(v, 0)\right) = G(2p - \beta\Delta w) \tag{62}$$

$$q_1 = \Pr\left(u_1(v, 1) > u_0(v, 1)\right) = G(\beta\Delta w). \tag{63}$$

The state transitions also differ from those in the benchmark case. Let (r_0, r_1) be the steady-state credit distributions:

$$r_0 = (1 - \lambda) \cdot (r_0(1 - q_0) + r_1 q_1) \tag{64}$$

$$r_1 = \lambda + (1 - \lambda) \cdot (r_0 q_0 + r_1(1 - q_1)). \tag{65}$$

The first equality means that those without a credit must have just previously experienced a leisure occasion and either not consumed with $s = 0$ or consumed with $s = 1$. The second equality means that those with a credit must have either just previously experienced an employer-paid occasion or a leisure occasion in which they either consumed with $s = 0$ or did not consume with $s = 1$. Solving the system of two equations:

$$r_0 = \frac{q_1}{q_1 + q_0 + l} \tag{66}$$

$$r_1 = \frac{q_0 + l}{q_1 + q_0 + l}, \tag{67}$$

where

$$l = \frac{\lambda}{1 - \lambda}. \tag{68}$$

Because of the employer-paid occasions, the purchase amount from "high-price" customers exceeds that from "low-price" customers even though the market is in a steady state. Letting D_H and D_L represent purchases at the two price levels, respectively:

$$D_H = \lambda + (1 - \lambda) \cdot r_0 q_0 = \frac{l(q_0 + q_1 + l) + q_0 q_1}{(1 + l)(q_0 + q_1 + l)}, \tag{69}$$

$$D_L = (1 - \lambda) \cdot r_1 q_1 = \frac{l q_1 + q_0 q_1}{(1 + l) \cdot (q_0 + q_1 + l)}, \tag{70}$$

and it is easy to show that $D_H > D_L$.

The profit from the reward program is thus:

$$\pi_R(p) = \lambda \cdot (2p) + (1-\lambda)\cdot(2p)\cdot r_0 q_0 \tag{71}$$

$$= \lambda \cdot (2p) + (1-\lambda)\cdot p \cdot \frac{2q_0 q_1}{q_0+q_1+l}. \tag{72}$$

Compared with the profit under spot pricing in Equation (57), the reward program yields higher profit on employer-paid occasions by charging a price $p_0 = 2p$ rather than p, but a lower profit on leisure occasions as:

$$Ip \cdot \frac{2q_0 q_1}{q_0+q_1+l} < p \cdot \frac{2q_0 q_1}{q_0+q_1} \leq p \cdot G(p). \tag{73}$$

When higher profits on employer-paid occasions dominate the lower profits on leisure occasions, the reward program can outperform spot pricing. This occurs when λ is high so that business travel is frequent, and β is high so that q_0 and q_1 do not differ by much (see footnote 9), and the average demand of type-0 and type-1 customers is not that different from demand under spot pricing. We illustrate this by providing a concrete example in a B1G1 reward program with uniformly-distributed valuations:

Proposition 5. *In a B1G1 program with $v \sim U(0,1)$, for any $v_2 > 1$ and $l < \bar{l}(v_2)$, there exists $\bar{\beta} \in (0,1)$, such that for any $\beta > \bar{\beta}$, firms can earn higher profits in the reward program by charging $p_0 = 1$:*

$$\pi_R(0.5) \geq \max_p \pi_S(p). \tag{74}$$

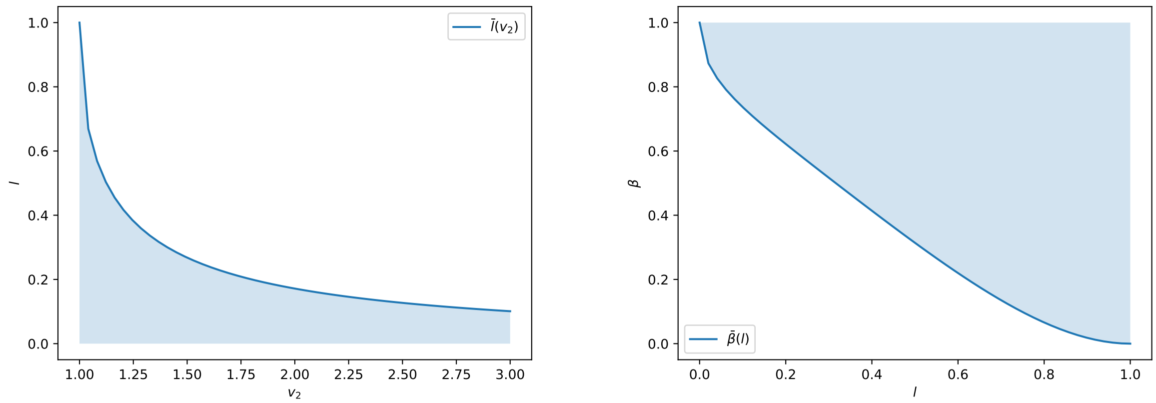
The two cutoffs, $\bar{l}$ and $\bar{\beta}$ are:

$$\bar{l} = 2v_2 - 1 - \sqrt{(2v_2-1)^2 - 1} \tag{75}$$

$$\bar{\beta} = (l+1)\left(1 - \sqrt{l + l^2 - l^3}\right). \tag{76}$$

The left panel of Figure 3 plots the restriction on l, and thus λ, as a function of v_2. This restriction arises because firms can always charge a price v_2 and sell only on employer-paid occasions, and the two programs generate the same profit. To rule out such unrealistic cases, the probability for employer-paid occasions must be sufficiently low, especially when v_2 is high. Specifically, we require $l < \bar{l}(v_2)$, which is decreasing in v_2. The right panel of Figure 3 shows the constraints on (l, β). The constraints do not depend on v_2 because if firms serve leisure occasions, then $p < v_2$ for sure, and the profit on employer-paid occasions does not depend on v_2. Reward program profits are more likely to exceed those under spot pricing when

Figure 3: Comparison of reward program and spot pricing profits with third-party payment



Notes: Conditions under which the reward program, with valuation distribution $U(0,1)$, generates higher profits than spot pricing: $l < \bar{l}(v_2)$ and $\beta > \bar{\beta}(l)$.

employer-paid occasions are more frequent (l is higher). This is intuitive, as the customer always purchases on employer-paid occasions. Finally, when the discount factor β is higher, the reward program also is more profitable because the realized demands q_0 and q_1 are closer, which makes their average closer to the realized demand under spot pricing.

Because demand is exogenously segmented (employer-paid and leisure occasions), having two prices in the reward program allows firms to price discriminate. The firm can charge a higher price on work occasions (when demand is less elastic) and credit-earning leisure occasions (when valuation is high) and a lower price on credit-redeeming leisure occasions (when valuation is low). Spot pricing lacks this flexibility, imposing a tradeoff on the firm: price high and lose more leisure occasions, or price low and give up profits on employer-paid occasions. The reward program alleviates this trade-off by decoupling the prices for these segments.

4.3 Finite expiration

The third exception is finite-credit expiration. In this case, the harmonic mean of the purchase probabilities in the reward program exceeds the purchase probability under spot pricing, even though the market is in a steady state and the number of purchases by type-0 and type-1 customers are equal.[11] Imposing a finite-credit expiration endogenously raises the number of purchases by smoothing out demand and alleviating the

[11] The equality of purchases property could also be violated if not all type-1 customers redeem (i.e., v is below zero for some customers rather than above zero for all customers as we assume in our example).

price-sales tradeoff. To illustrate, consider a B1G1 program with credits that expire in one period. Such a reward program can be interpreted as "bundling" two products: one "spot" product consumed today and one "future" product consumed tomorrow. As in classic bundled pricing (Adams and Yellen, 1976), the firm can increase profits relative to pricing the products separately if it reduces the variance of the demand distribution. In this case, the "future" product makes ex-post customers more similar ex-ante so that the firm does not face as extreme a trade-off between pricing high and losing low-valuation customers versus pricing low and getting more demand.

We consider a B1G1 program with $p_0 = 2p$ and $p_1 = 0$, and assume credits expire in one period. Let $u_a(v, s)$ be the payoff of taking action $a \in \{0, 1\}$ with $s \in \{0, 1\}$ credits:

$$u_1(v, s) = v - p_s + \beta \cdot w(1 - s) \tag{77}$$

$$u_0(v, s) = \beta \cdot w(0), \tag{78}$$

where the value function is determined as:

$$w(s) = E \max \left(u_1(v, s), u_0(v, s)\right). \tag{79}$$

This differs from the benchmark case in that the payoff from waiting if the customer holds one credit (Equation (78)), $u_0(v, 1)$, is $\beta \cdot w(0)$ rather than $\beta \cdot w(1)$ due to the credit expiration. Next, we solve for the purchase probabilities. For simplicity, we assume the utility from consumption is positive:

$$v > 0. \tag{80}$$

When $s = 1$, customers will consume for sure ($u_1(v, 1) > u_0(v, 1)$), as the credit will expire if not used, and the purchase likelihood is:

$$q_1 = 1 = G(0). \tag{81}$$

When $s = 0$, the purchase likelihood is:

$$q_0 = \Pr(v - p_0 + \beta w(1) \geq \beta w(0)) = G(p_0 - \beta \Delta w). \tag{82}$$

The steady-state distribution of credits (r_0, r_1) is determined similarly as in the benchmark case:

$$r_0 = \frac{q_1}{q_0 + q_1} = \frac{1}{q_0 + 1} \tag{83}$$

$$r_1 = \frac{q_0}{q_0 + q_1} = \frac{q_0}{q_0 + 1}. \tag{84}$$

The equality of purchase amounts across the two states holds in this case:

$$r_0 q_0 = r_1 q_1. \tag{85}$$

Applying the notation of the benchmark model (with $\delta = p$), the profit for the reward program simplifies to:

$$\pi_R(p) = (p + p) \cdot (r_0 q_0) + (p - p) \cdot (r_1 q_1) = p \cdot \frac{2 q_0 q_1}{q_0 + q_1} = p \cdot Q(p, p). \tag{86}$$

What is different from the benchmark case in Proposition 1 is that the third observation (related to average purchase probability) does not hold even with the "not too convex" assumption. To see this, we know $q_0 = G(2p - \beta \Delta w)$, $q_1 = G(0)$, and the convexity of the reciprocal of demand only guarantees that:

$$Q(p, p) \leq G\left(p - \frac{\beta}{2} \Delta w\right). \tag{87}$$

However, $Q(p, 0) = G(p)$, which can be smaller than $Q(p, p)$.

To illustrate this, we provide an example with a uniform distribution of valuations:

Proposition 6. *For a B1G1 program with $v \sim U(0,1)$ and credit expiration of one period, when $\beta > \frac{3}{4}$, $\pi_R(0.5) \geq \pi_S^* = \frac{1}{4}$.*

Although we are not the first to propose that finite-credit expiration may result in a reward program outperforming spot pricing in a non-behavioral setting, our explanation differs from previous studies (Sun and Zhang, 2019; Liu et al., 2021). The reward program with one-period expiration can be regarded as a bundle of two products: one "spot" product consumed today and one "future" product consumed tomorrow. If the firm charges separately for the two products, the optimal "spot" price ($p = 0.5$) maximizes:

$$\pi_S(p) = p \cdot G(p) = p \cdot (1 - p), \tag{88}$$

yielding sales of $1/2$, and profits of $\pi_S^* = 1/4$.

On the other hand, the optimal price for the "future" product is $p = \beta/2$. As customers make purchase decisions based on their discounted expected valuation ($\beta \cdot E(v) = \beta/2$), demand is more uniform and the firm can expand sales. It follows immediately that when the discount factor β is sufficiently high ($\beta > 0.5$), purchasing the future product under the reward program can generate higher profits than spot pricing. Since the reward program is a combination of a spot and future product, when $\beta > \frac{3}{4}$ (Proposition 6) it will be more profitable compared to the period-by-period selling of spot products. In other words, this smoothing of future demand across customers is more important the higher the customers' discount factor (as demonstrated in Proposition 6), since the present value of the gains from bundling is higher.

Finally, why does this argument not hold for our baseline example with no expiration? In the finite-expiration case, customers may suffer in the future period when they consume the product: if their valuation turns out to be low, the value they obtain is lower than the price they paid in advance. That is, they over-consume ex-post. However, with no expiration, credits can be used any time in the future, and customers will wait to redeem their credit rather than over-consuming if their realized valuation is low. The finite expiration is necessary to enforce the "bundling" of the products.

5 Conclusion

In this paper, we consider the incentive for a monopolist to offer a frequency reward program. Most previous studies focus on the effects of a reward program, assuming its adoption. The few papers that examine the incentive to offer a program use somewhat specific assumptions on customer preferences, market participation, or customers' time horizons. We consider a general setting that nests standard empirical specifications, and find that spot pricing dominates. Although we consider a reward program which offers price discounts based on previous cumulative purchases, our analysis also applies to a prepayment program in which customers receive a discount for purchasing above a certain volume but consume some units in the future.[12]

Using this as a point of departure, we examine how the reward program's environment or design can be modified to yield higher profits than spot pricing. We identify specific conditions required for this to hold and offer three examples. There may be other examples in which a reward program can outperform spot pricing in a non-behavioral setting; however, they would need to violate one or more of the three conditions we identify. This would be useful to examine as future work.

Although we illustrate how the reward program can outperform spot pricing in three cases, further work

[12] Li et al. (2023) analyze such a prepayment program in a two-period model.

is needed on how to optimize the program design under these exceptions. We provided examples with a fixed price. It would be useful to examine the optimal price of the reward program under these exceptions. There is also further work to be done on the specific exceptions. For a new product introduction, whether firms can commit to a fixed price over time and, if not, how to dynamically set prices. In the context of third-party payment, whether to allow customers to accumulate more than one credit and for finite-credit expiration, how to optimally set the duration of credit expiration.

It would also be useful to extend the model to examine behavioral settings. Several settings could be incorporated into the model relatively easily to determine whether they affect reward program profitability relative to spot pricing. For example, additional utility from consumption via points redemption versus via purchase, and "slippage" from customers failing to redeem earned credits. These would require minimal changes to the model and could be incorporated while preserving its analytical tractability. Other settings may require more extensive modifications.

There are design features or settings that we do not consider. It would be useful to see if our results for in-kind rewards translate to a setting with monetary rewards. For the multiple-credit program, it would be useful to examine how to optimally set the thresholds (i.e., X in a BXG1 program). We also do not consider reward programs not controlled by the seller (e.g., a franchisee as in Chung et al., 2022), reward accumulation based on monetary rather than quantity purchases (Chun and Ovchinnikov, 2019), and coalitions of reward programs (Lederman, 2007).

Our results suggest the need for additional empirical research on the profitability of offering a reward program rather than examining the effects of an already-established program. There has been more empirical work on the latter than the former. Our results also suggest empirically examining how changing from no-expiration to finite-expiration of credits, or vice versa, affects the profitability of a reward program.

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A Proofs

A.1 Proof of Proposition 2

First, solve the steady-state distribution $\{r_s\}$ in the generalized BYGZ program, with X = Y + Z:

$$r_0 = q_X r_X + (1 - q_0) r_0 \tag{A1}$$
$$r_1 = q_0 r_0 + (1 - q_1) r_1 \tag{A2}$$
$$\dots \tag{A3}$$
$$r_{X-1} = q_{X-2} r_{X-2} + (1 - q_{X-1}) r_{X-1}, \tag{A4}$$

where

$$q_s = G(p_s - \beta \Delta w(s)). \tag{A5}$$

This is a system of linear equations, which has one unique solution:

$$r_s = \frac{1}{q_s} \cdot \left(\sum_{s=0}^{X-1} \frac{1}{q_s} \right)^{-1}. \tag{A6}$$

Consequently, the demand function:

$$D_{RX}(p) = \sum_{s=0}^{X-1} r_s \cdot q_s = X \cdot \left(\sum_{s=0}^{X-1} \frac{1}{q_s} \right)^{-1}. \tag{A7}$$

Following similar logic to that in Proposition 1, since $\sum_s \Delta w(s) = 0$ and $H(\cdot)$ is convex, we can apply Jensen's inequality to get:

$$\frac{1}{X} \left(\sum_{s=0}^{X-1} H(p_s - \beta \Delta w(s)) \right) \geq \frac{1}{X} H \left(\sum_{s=0}^{X-1} p_s \right). \tag{A8}$$

Plugging back into Equations (A5) and (A7), spot pricing demand exceeds reward program demand.

A.2 Proof of Lemma 1

By induction. It suffices to show that for $r > \bar{r} = \frac{q_1}{q_0 + q_1}$, and:

$$r' = q_1 + (1 - q_0 - q_1) \cdot r, \tag{A9}$$

we have $\bar{r} < r' < r$.

To show $r' > \bar{r}$, we have $r' > q_1 + (1 - q_0 - q_1) \cdot \bar{r} = \bar{r}$, using the fact that $\bar{r} = \frac{q_1}{q_0 + q_1}$.

To show $r' < r$, we have $r' - r = q_1 - (q_1 + q_0) \cdot r < q_1 - (q_1 + q_0) \cdot \bar{r} = 0$ again using the fact that $\bar{r} = \frac{q_1}{q_0 + q_1}$.

So $\{r_{0t}\}$ is a bounded and decreasing sequence and therefore must have a limit. The lower limit, using the fact that $\bar{r} = \frac{q_1}{q_0 + q_1}$, is derived by setting $r = q_1 + (1 - q_0 - q_1) \cdot r$.

A.3 Proof of Proposition 4

We first analytically solve for the value function when $v \sim U(0, 1)$. We then calculate the profit for the reward program, and compare that with the optimal profit from static pricing to prove the result. We only need to focus on a classic B1G1 program with $p_0 = 1$ and $p_1 = 0$.

A.3.1 Solving for value function

The value function is determined by the systems of equations in (3) and (4):

$$w_0 = E \max (v - p_0 + \beta w_1, \beta w_0) \tag{A10}$$
$$w_1 = E \max (v + \beta w_0, \beta w_1) . \tag{A11}$$

Re-organizing the terms in both equations:

$$(1-\beta) w_0 = E \max (v - p_0 + \beta \Delta w, 0) \tag{A12}$$
$$(1-\beta) w_1 = E \max (v - \beta \Delta w, 0) , \tag{A13}$$

where $\Delta w = w_1 - w_0$.

The following property is helpful in calculating the value function.

Lemma A.2. *For any $p_0 > 0$, the value function specified from above has the following property:*

$$0 \leq \beta \Delta w \leq \frac{p_0}{2}. \tag{A14}$$

Proof. We prove by contradiction. First, suppose $\beta \Delta w < 0$, then $w_1 < w_0$, and the right-hand side of (A13) is weakly larger than the right-hand side of (A12), meaning $w_1 \geq w_0$, contradiction. So $\beta \Delta w \geq 0$. Second, suppose $\beta \Delta w > \frac{p_0}{2}$, then $p_0 - \beta \Delta w < \beta \Delta w$, and the right-hand side of (A13) is weakly smaller than the right-hand-side of (A12), meaning $w_1 \leq w_0$. Then $w_1 = w_0$, which contradicts $\beta \Delta w > \frac{p_0}{2} > 0$. □

For a uniformly-distributed random variable, the following additional property is helpful:

Lemma A.3. *For $v \sim U(0,1)$:*

$$E \max (v - x, 0) = \begin{cases} \frac{1}{2}(1-x)^2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x > 1. \end{cases} \tag{A15}$$

Proof. For $0 \leq x \leq 1$:

$$E \max(v - x, 0) = \Pr(v > x) \cdot E\left(v - x | x \leq v \leq 1\right) = (1-x) \cdot \left(\frac{1}{2}(1-x)\right) = \frac{1}{2}(1-x)^2. \tag{A16}$$

□

We calculate the value function, by taking the difference of the two equations (A12) and (A13) and write:

$$t = \beta \Delta w, \tag{A17}$$

we have:

$$\left(\frac{1}{\beta} - 1\right) \cdot t = E \max (v - t, 0) - E \max (v - p_0 + t, 0) . \tag{A18}$$

As $p_0 = 1$, $0 \leq t \leq \frac{1}{2}$ given Equation (A14), applying Lemma A.3 and plugging in $p_0 = 1$:

$$\left(\frac{1}{\beta} - 1\right) \cdot t = \frac{1}{2}(1-t)^2 - \frac{1}{2}t^2 = \frac{1}{2} - t, \tag{A19}$$

so

$$t = \beta \Delta w = \frac{\beta}{2}. \tag{A20}$$

A.3.2 Solving for the profits

Since $\beta\Delta w = \frac{\beta}{2}$ and $p_0 = 1$, $q_0 = G(p_0 - \beta\Delta w) = \frac{\beta}{2}$ and $q_1 = G(\beta\Delta w) = 1 - \frac{\beta}{2}$ from Equations (49) and (50). In this case, $q_0 + q_1 = 1$ and customers' credits reach the steady-state in $t = 2$, i.e., $r_0(2) = q_1 = \bar{r}$.

Note that the per-period profit in (51) for $r_0(t) = r_0$ is:

$$\pi(p, r_0) = (2p) \cdot r_0 \cdot q_0, \tag{A21}$$

and in this case, $p = \frac{p_0}{2} = \frac{1}{2}$, $q_0 = 1 - \frac{\beta}{2}$, $r_0(1) = 1$, and $r_0(t) = \bar{r} = q_1$ for $t > 1$. So, using Equation (51):

$$\pi(p, 1) = q_0 = \frac{\beta}{2} \tag{A22}$$

$$\pi(p, \bar{r}) = q_0 q_1 = \frac{\beta}{2}\left(1 - \frac{\beta}{2}\right). \tag{A23}$$

So the profit for the reward program is:

$$\Pi_R = \pi(p, 1) + \gamma \cdot \frac{\pi(p, \bar{r})}{1 - \gamma} \tag{A24}$$

$$= \frac{\beta}{2} \cdot \left(1 + \frac{\gamma}{1 - \gamma} \cdot \left(1 - \frac{\beta}{2}\right)\right). \tag{A25}$$

On the other hand, the single-period static profit is:

$$\pi_s = \max_p p \cdot G(p) = \max_p p \cdot (1 - p) = \frac{1}{4}, \tag{A26}$$

so the total profit is:

$$\Pi_S = \frac{\pi_s}{1 - \gamma} = \frac{1}{4(1 - \gamma)}. \tag{A27}$$

For the reward program to be more profitable, we solve for the inequality

$$\Pi_R > \Pi_S, \tag{A28}$$

which gives

$$\gamma < \frac{2\beta - 1}{\beta^2}. \tag{A29}$$

So the reward program charging $p_0 = 1$ and $p_1 = 0$ can out-perform the static program if $\beta > \frac{1}{2}$ and $\gamma < \frac{2\beta - 1}{\beta^2}$.

A.4 Proof of Proposition 5

To show that firms can earn a higher profit by offering a reward program with $p_0 = 1$ when the valuation is uniformly distributed and there are employer-paid occasions, we proceed in three steps: derive the optimal profit under static pricing, derive the optimal profit under reward program with $p_0 = 1$, and compare the two.

A.4.1 Static pricing

We first solve for static pricing:

Proposition A.7. *Let $l = \frac{\lambda}{1-\lambda}$, and $\bar{l} \in (0, 1)$ solves:*

$$(\bar{l} + 1)^2 - 4v_2 \cdot \bar{l} = 0, \tag{A30}$$

That is:

$$\bar{l}(v_2) = 2v_2 - 1 - \sqrt{(2v_2-1)^2 - 1}. \tag{A31}$$

When $v \sim U(0,1)$ and $v_2 > 1$, the optimal static pricing and profits are:

1. *When $0 < l \leq \bar{l}$, both personal and business users will buy, and*

$$p_S^* = \frac{1}{2}(l+1) \tag{A32}$$

$$\pi_S^* = \frac{1}{4}(l+1). \tag{A33}$$

2. *When $l > \bar{l}$, only business users will buy, and*

$$p_S^* = v_2 \tag{A34}$$

$$\pi_S^* = \lambda v_2. \tag{A35}$$

Proof. First, we show

$$\bar{l} < 1 \tag{A36}$$

In fact, when $v_2 > 1$, using the definition in Equation (A31):

$$\bar{l} = \frac{\left(2v_2 - 1 - \sqrt{(2v_2-1)^2-1}\right)\cdot\left(2v_2 - 1 + \sqrt{(2v_2-1)^2-1}\right)}{2v_2 - 1 + \sqrt{(2v_2-1)^2-1}} \tag{A37}$$

$$= \frac{1}{2v_2 - 1 + \sqrt{(2v_2-1)^2-1}} < 1. \tag{A38}$$

From (57), the profit for static pricing is:

$$\pi(p) = \lambda \cdot p + (1-\lambda)\cdot p \cdot G(p) \tag{A39}$$

$$= \begin{cases} p\cdot(\lambda + (1-\lambda)(1-p)) & \text{if } p \in (0,1) \\ \lambda \cdot p & \text{if } p \in [1, v_2) \\ 0 & \text{if } p \geq v_2. \end{cases} \tag{A40}$$

Since $l = \frac{\lambda}{1-\lambda}$, $\frac{1}{1-\lambda} = l+1$. For $p \in (0,1)$, both leisure and employer-paid occasions occur, and the corresponding profit is defined as:

$$\pi_{s1} \max_p p\cdot(\lambda + (1-\lambda)(1-p)) \tag{A41}$$

$$= (1-\lambda)\cdot \max_p p \cdot \left(\frac{\lambda}{1-\lambda} + 1 - p\right) \tag{A42}$$

$$= \frac{1}{l+1}\cdot \max_p p\cdot(l+1-p) \tag{A43}$$

$$= \begin{cases} \frac{1}{l+1}\cdot\frac{(l+1)^2}{4} = \frac{l+1}{4} & \text{if } \frac{l+1}{2} < 1 \\ \frac{l}{l+1} & \text{if } \frac{l+1}{2} \geq 1. \end{cases} \tag{A44}$$

So the optimal profit for static pricing is:

$$\pi_S^* = \max_p \pi(p) = \max\left(\pi_{s1}, \lambda v_2\right). \tag{A45}$$

When $l < 1$, $\pi_{s1} = \frac{l+1}{4}$, and:

$$\pi_S^* = \max\left(\frac{l+1}{4}, \frac{l}{l+1}v_2\right) = \begin{cases} \frac{l+1}{4} & \text{if } l \leq \bar{l} \\ \frac{l}{l+1}v_2 & \text{if } \bar{l} < l < 1, \end{cases} \tag{A46}$$

where the second equality comes from (A30) and the fact that $\bar{l} < 1$. When $l \geq 1$, $\pi_{s1} = \frac{l}{l+1}$, and:

$$\pi_S^* = \max\left(\frac{l}{l+1}, \frac{l}{l+1}v_2\right) = \frac{l}{l+1}v_2. \tag{A47}$$

So if $l \leq \bar{l}$, $p_S^* = \frac{l+1}{2}$ and $\pi_S^* = \frac{l+1}{4}$. When $l > \bar{l}$, $p_S^* = v_2$ and $\pi_S^* = \frac{l}{l+1}v_2 = \lambda v_2$. □

A.4.2 B1G1 program with $p_0 = 1$

We now solve for the profit of the reward program. We first prove a property for the value function in a similar manner as in Lemma A.2, and then calculate the profit.

Following (61), the value function in this case is determined as:

$$w_0 = \lambda\beta w_1 + (1-\lambda)\cdot E\max\left(v - p_0 + \beta w_1, \beta w_0\right) \tag{A48}$$
$$w_1 = \lambda\beta w_1 + (1-\lambda)\cdot E\max\left(v + \beta w_0, \beta w_1\right). \tag{A49}$$

Re-organizing the terms:

$$(1-(1-\lambda)\cdot\beta)w_0 = \lambda\beta w_1 + (1-\lambda)\cdot E\max\left(v - p_0 + \beta\Delta w, 0\right) \tag{A50}$$
$$(1-(1-\lambda)\cdot\beta)w_1 = \lambda\beta w_1 + (1-\lambda)\cdot E\max\left(v - \beta\Delta w, 0\right). \tag{A51}$$

Following the same logic as Lemma A.2 but using Equations (A50) and (A51) rather than (A12) and (A13), we can show:

$$0 \leq \beta\Delta w \leq \frac{p_0}{2}. \tag{A52}$$

Write $t = \beta\Delta w \in (0,1)$ as $p_0 = 1$, take the difference between the above two equations:

$$(\frac{1}{\beta} - (1-\lambda))\cdot t = (1-\lambda)\cdot\left(E\max(v-t,0) - E\max\left(v-1+t,0\right)\right) \tag{A53}$$
$$= \frac{1-\lambda}{2}\cdot\left((1-t)^2 - t^2\right) = (1-\lambda)\cdot\left(\frac{1}{2} - t\right), \tag{A54}$$

where the calculation of $E\max(\cdot)$ uses Lemma A.3. So, solving the above two equations:

$$t = \beta\Delta w = \frac{\beta(1-\lambda)}{2} = \frac{\beta}{2(l+1)}. \tag{A55}$$

With the value function, the purchase probabilities in (62) and (63) are (since $p_0 = 1$):

$$q_0 = G(p_0 - t) = t \tag{A56}$$
$$q_1 = G(t) = 1 - t, \tag{A57}$$

and the profit for B1G1 in (72) is thus

$$\pi_R(0.5) = \lambda + (1-\lambda)\cdot\frac{q_0 q_1}{q_0 + q_1 + l} = \frac{l}{1+l} + \frac{t(1-t)}{(1+l)^2}. \tag{A58}$$

A.4.3 Comparing the two profit levels

Under static pricing, depending on the ratio of employer-paid occasions (i.e., l), firms may optimally: serve both types of occasions if l is not too high, or sell only on employer-paid occasions if l is high enough. We consider only the former as it is more realistic (i.e., $l < \bar{l}(v_2)$). We now compare the two profit levels:

Proposition A.8. *When $l < \bar{l}(v_2)$, if*

$$(1+l)\cdot\left(1-\sqrt{l+l^2-l^3}\right) < \beta < 1, \tag{A59}$$

we have $\pi_R(0.5) > \pi_S^*$.

Proof. First, we show $(1+l)\cdot\left(1-\sqrt{l+l^2-l^3}\right) < 1$. Write

$$c = 1-(l+l^2-l^3) = (1-l)^2(1+l) = (1-l^2)(1-l) < 1, \tag{A60}$$

so:

$$\begin{aligned} (1+l)\cdot\left(1-\sqrt{l+l^2-l^3}\right) &= (1+l)(1-\sqrt{1-c}) && \text{(A61)}\\ &= \frac{(1+l)\cdot c}{1+\sqrt{1-c}} = \frac{(1+l)^2(1-l)^2}{1+\sqrt{1-c}} && \text{(A62)}\\ &< (1-l^2)^2 < 1. && \text{(A63)} \end{aligned}$$

Second, we show $\pi_R > \pi_S^* = \frac{1+l}{4}$ (the last equality comes as $l < \bar{l}(v_2)$ that firms serve both types):

$$\frac{l}{1+l} + \frac{t(1-t)}{(1+l)^2} > \frac{1+l}{4}. \tag{A64}$$

Rearranging:

$$\frac{t(1-t)}{(1+l)^2} > \frac{l+1}{4} - \frac{l}{1+l} = \frac{(1-l)^2}{4(1+l)}, \tag{A65}$$

or:

$$4t(1-t) > (1+l)(1-l)^2. \tag{A66}$$

Using the fact that $c = (1+l)(1-l)^2$ in Equation (A60):

$$4t(1-t) > c. \tag{A67}$$

Define

$$f(t) = 4t^2 - 4t + c, \tag{A68}$$

Recall that $t = \frac{\beta}{2(1+l)} \in (0, \frac{1}{2})$ and note that $c \in (0,1)$. As $f(0) = c > 0$, $f(\frac{1}{2}) = -1 + c < 0$, and $f'(t) = 8t - 4 > 0$ to show $f(t) < 0$, it suffices to show $t > t_1$, where t_1 is the smaller root of $f(t) = 0$:

$$t_1 = \frac{4-\sqrt{16-16c}}{8} = \frac{1-\sqrt{1-c}}{2}. \tag{A69}$$

Finally, as $\beta > (1+l)\cdot(1-\sqrt{1-c})$ (from Equation (A59)), $t = \frac{\beta}{2(1+l)} > \frac{1-\sqrt{1-c}}{2}$, which completes the proof. □

A.5 Proof of Proposition 6

We first solve for the value function, and then the profit.

A.5.1 Value function

The value function when the credit expires in one period is:

$$w_0 = E \max(v - p_0 + \beta w_1, \beta w_0) \tag{A70}$$

$$w_1 = E \max(v + \beta w_0, \beta w_0) = E(v) + \beta w_0. \tag{A71}$$

Re-organizing both equations, we have:

$$(1-\beta)w_0 = E \max(v - p_0 + \beta \Delta w, 0) \tag{A72}$$

$$(1-\beta)w_1 = E \max(v - \beta \Delta w, -\beta \Delta w) \tag{A73}$$

We first show the following property.

Lemma A.4. *The value function specified from above satisfies*

$$0 \leq \beta \Delta w \leq \frac{p_0}{2}. \tag{A74}$$

Proof. We prove by contradiction. First suppose $\Delta w < 0$, then the right-hand side of (A73) is larger than that of (A72), meaning $w_1 \geq w_0$, contradiction. So $\Delta w \geq 0$. Second, suppose $\beta \Delta w > \frac{p_0}{2}$, then $p_0 - \beta \Delta w < \beta \Delta w$, and the right-hand-side of (A72) is larger than that of (A73), impling $w_0 \geq w_1$. Thus $w_0 = w_1$, which contradicts $\beta \Delta w > \frac{p_0}{2}$. □

We fix $p_0 = 1$. As $v \sim U(0,1)$, $E(v) = \frac{1}{2}$. Since $\beta \Delta w \leq \frac{p_0}{2} = \frac{1}{2}$, the two value functions are:

$$(1-\beta)w_0 = E \max(v - 1 + \beta \Delta w, 0) \tag{A75}$$

$$(1-\beta)w_1 = \frac{1}{2} - \beta \Delta w. \tag{A76}$$

Let $t = \beta \Delta w$, take the difference between the two, we have:

$$\left(\frac{1}{\beta} - 1\right) t = \frac{1}{2} - t - \frac{1}{2}t^2, \tag{A77}$$

where the last term comes from the fact that $\beta \Delta w \in (0, \frac{1}{2})$, and Lemma A.3. So t is determined by $f(t) = 0$, where:

$$f(t) = \frac{1}{2}t^2 + \frac{1}{\beta}t - \frac{1}{2}. \tag{A78}$$

Finally, as $f(0) < 0$, $f(\frac{1}{2}) = \frac{1}{8} + \frac{1}{2\beta} - \frac{1}{2} > \frac{1}{8}$, and $f'(t) > 0$, there exists one root in $\left(0, \frac{1}{2}\right)$.

A.5.2 Profit

Fix $p_0 = 1$, and $t = \beta \Delta w$. Then from (82) and (81), $q_0 = G(p_0 - \beta \Delta w) = t$, $q_1 = 1$, and the profit for the finite-expiration reward program is:

$$\pi_R = \frac{q_0 q_1}{q_0 + q_1} = \frac{t}{1+t}. \tag{A79}$$

For the reward program to exceed the static one:

$$\pi_R > \pi_S^* = \frac{1}{4}, \tag{A80}$$

it is equivalent to have:

$$t > \frac{1}{3}. \tag{A81}$$

As we have shown, $f\left(\frac{1}{2}\right) > \frac{1}{8} > 0$ and $f' > 0$, in order to have a root between $\frac{1}{3}$ and $\frac{1}{2}$, we must have:

$$f(\frac{1}{3}) < 0, \tag{A82}$$

Plug in Equation (A78):

$$\frac{1}{18} + \frac{1}{3\beta} - \frac{1}{2} < 0 \tag{A83}$$

which gives

$$\beta > \frac{3}{4}. \tag{A84}$$

B Distributions Fulfilling Assumption 1 (Reciprocal of Survival Function is Convex)

This appendix catalogs common distributions that meet Assumption 1 (reciprocal of the survival function is convex). This is a weak form of log-concavity of a distribution and is met by most commonly-used distributions.

B.1 Definitions

Probability density function:

$$f\left(x\right) \tag{B.1}$$

Cumulative density function:

$$F\left(x\right) \tag{B.2}$$

Survival function:

$$G\left(x\right) = 1 - F\left(x\right) \tag{B.3}$$

B.2 Conditions

Assumption 1 in the main text (reciprocal of the survival function is convex) is the weakest of three types of log-concavity for distributions. We examine all three types since if one of the stricter types of log-concavity is met, then the weaker forms are also met and these are sometimes more straightforward to show. The three types of log-concavity in decreasing order of strictness are:

Condition 1: Concavity of survival function: $G\left(x\right)$ is concave ($F\left(x\right)$ is convex) in x, which requires:

$$f'\left(x\right) > 0. \tag{B.4}$$

Condition 2: Log-concavity of survival function: $log\left(G\left(x\right)\right)$ is concave, which implies $F\left(x\right)$ has a hazard rate:

$$h\left(x\right) = \frac{f\left(x\right)}{1 - F\left(x\right)}, \tag{B.5}$$

that is increasing in x:

$$h'\left(x\right) > 0. \tag{B.6}$$

Condition 3 (corresponding to Assumption 1 in the main text): Reciprocal of the survival function is convex. $w\left(x\right) = G\left(x\right)^{-1} = \frac{1}{1-F(x)}$ is convex:

$$w''\left(x\right) = \frac{f'\left(x\right)}{\left(1 - F\left(x\right)\right)^2} + \frac{f\left(x\right)^2}{\left(1 - F\left(x\right)\right)^3} > 0, \tag{B.7}$$

or:

$$f'\left(x\right)\left(1 - F\left(x\right)\right) + f\left(x\right)^2 > 0. \tag{B.8}$$

B.3 Distributions

We verify Assumption 1 is met for the following distributions:

1. Uniform
2. Exponential
3. Normal
4. Beta
5. Pareto
6. Generalized Extreme value (Weibull distribution)
7. Chi-squared

B.4 Checking Conditions

B.4.1 Uniform Distribution

The pdf is:

$$f\left(x;u,l\right)=\begin{cases}\frac{1}{u-l} & \text{for } x\in[l,u]\\ 0 & \text{otherwise.}\end{cases} \tag{B.9}$$

The cdf is:

$$F\left(x;u,l\right)=\begin{cases}0 & \text{for } x<l\\ \frac{x-l}{u-l} & \text{for } x\in[l,u]\\ 1 & \text{for } x>u.\end{cases} \tag{B.10}$$

The hazard function is defined on $[l,u]$ as:

$$h\left(x;u,l\right)=\frac{1}{u-x}. \tag{B.11}$$

Condition 1 is not met:

$$f'\left(x;u,l\right)=0. \tag{B.12}$$

But Condition 2 is met:

$$h'\left(x;u,l\right)=\frac{1}{\left(u-x\right)^2}>0. \tag{B.13}$$

B.4.2 Exponential Distribution

The pdf is:

$$f\left(x;\lambda\right)=\lambda exp\left(-\lambda x\right), \tag{B.14}$$

where $\lambda>0$. The cdf is:

$$F\left(x;\lambda\right)=1-exp\left(-\lambda x\right). \tag{B.15}$$

The hazard function is:

$$h\left(x;\lambda\right)=\lambda. \tag{B.16}$$

The reciprocal of the survival function is:

$$w\left(x;\lambda\right)^{-1}=exp\left(\lambda x\right). \tag{B.17}$$

Condition 1 is not met:

$$f'\left(x;\lambda\right)=-\lambda^2 exp\left(-\lambda x\right)<0. \tag{B.18}$$

Condition 2 is also not met:

$$h'\left(x;\lambda\right)=0. \tag{B.19}$$

But Condition 3 is met:

$$w''(x; \lambda) = \lambda^2 exp(\lambda x) > 0. \tag{B.20}$$

B.4.3 Normal Distribution

The pdf is:

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}. \tag{B.21}$$

An alternative way to verify Condition 1 is to check if the log of the pdf function is concave (Bagnoli and Bergstrom (2005)). This confirms Condition 1:

$$log\left[f(x; \mu, \sigma)\right]'' = -\frac{1}{\sigma} < 0. \tag{B.22}$$

B.4.4 Beta Distribution

The pdf is:

$$f(x; \alpha, \beta) = \frac{1}{\mathcal{B}(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, \tag{B.23}$$

where $0 < x < 1$, $\mathcal{B} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, and Γ is the Gamma function.

Condition 1 is met if $\alpha > 1$ and $\beta > 1$ since:

$$f'(x; \alpha, \beta) = \frac{1}{\mathcal{B}(\alpha, \beta)}(\alpha-1)x^{\alpha-2}(1-x)^{\beta-1} + (\beta-1)x^{\alpha-1}(1-x)^{\beta-2} > 0. \tag{B.24}$$

B.4.5 Pareto Distribution

The pdf is:

$$f_X(x; \alpha, x_m) = \begin{cases} \frac{\alpha {x_m}^\alpha}{x^{\alpha+1}} & \text{if } x \geq x_m \\ 0 & \text{if } x < x_m, \end{cases} \tag{B.25}$$

where $\alpha > 0$ and $x_m > 0$. The cdf is:

$$F(x; \alpha, x_m) = \begin{cases} 1 - (\frac{x_m}{x})^\alpha & \text{if } x \geq x_m \\ 0 & \text{if } x < x_m. \end{cases} \tag{B.26}$$

The survival function is:

$$G(x; \alpha, x_m) = (\frac{x_m}{x})^\alpha. \tag{B.27}$$

The hazard function is:

$$h(x; \alpha, x_m) = \frac{\alpha}{x}, \tag{B.28}$$

where $x > x_m$. Condition 1 is not met since:

$$f'(x; \alpha, x_m) = -\alpha(\alpha+1)x_m^\alpha x^{-\alpha-2} < 0. \tag{B.29}$$

Condition 2 is not met since:

$$h'(x; \alpha, x_m) = -\alpha x^{-2} < 0. \tag{B.30}$$

However, Condition 3 is met since:

$$w''(x; \alpha, x_m) = \alpha(\alpha+1)x_m x^{-\alpha-2} > 0. \tag{B.31}$$

B.4.6 Generalized extreme value distribution (Weibull)

The pdf is:

$$f(x;c) = cx^{c-1}e^{-x^c}, \tag{B.32}$$

where $x > 0$ and $c \geq 1$. The cdf is:

$$F(x;c) = 1 - e^{-x^c}. \tag{B.33}$$

The survival function is:

$$G(x;c) = e^{-x^c}. \tag{B.34}$$

and the hazard rate is:

$$h(x;c) = cx^{c-1}. \tag{B.35}$$

Condition 1 is met only when $x < c - 1$:

$$f'(x;c) = c(c-1)x^{c-2}e^{-x^c} - cx^{2c-2}e^{-x^c} > 0. \tag{B.36}$$

However, Condition 2 is met for all values of x:

$$h'(x;c) = c(c-1)x^{c-2} > 0. \tag{B.37}$$

B.4.7 Chi-squared Distribution

The pdf is:

$$f(x;k) = \frac{1}{2^{\frac{k}{2}}}\Gamma(\frac{k}{2})x^{\frac{k}{2}-1}e^{-\frac{x}{2}}, \tag{B.38}$$

where Γ denotes the Gamma function and $k \geq 2$.

We can verify the log of the pdf function is concave to confirm Condition 1 (Bagnoli and Bergstrom (2005)):

$$log\,[f(x;k)]'' = -\left(\frac{k}{2} - 1\right)\frac{1}{x^2} < 0. \tag{B.39}$$