Does Speculative Activity Have Real Effects?

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JEL Classification: E32, G12, G18
Key Words: Heterogeneous belief, Production, Extraneous risk, Housing Boom and Bust, Speculation

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Abstract

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1 Introduction

The popular press often blames speculative activity in financial markets for impairing the real economy. Often times this accusation is accompanied by the suggestion that short horizon speculation also impacts the real economy. For example, institutional investors have recently begun to add commodity investments using futures as an alternative asset class. However, little theoretical work on this issue exists. In this paper, we develop new insights into how the real economy might interact with financial market activity.

An essential ingredient to generate trade is investor heterogeneity. Otherwise, if all investors are identical, prices which clear markets will imply no trade. Dumas (1989) models how heterogeneity in risk aversion affects equilibrium production decisions. To generate trade in our model, we assume investors have heterogeneous beliefs and time preferences. These sources of heterogeneity allow us to trace the impacts of short term versus long term speculative motives on the real economy and asset prices.

A large and growing literature has examined the effects of belief heterogeneity on asset prices.[1] in a pure exchange economy. However, these studies have fixed aggregate supply of consumption in each state and time. Therefore, this existing literature has characterized the effect of belief heterogeneity on discount rates.

However, the question of how production plans might be altered in response to these changes in discount rates has been less studied. Some literature on belief heterogeneity which does include productive decisions uses investors whose preferences are the sum of identically discounted expected logarithm of consumption at each date, see for example Detemple and Murthy (1994). When adjustment costs are not present, this is a limitation because these preferences will always choose the same production plans regardless of their beliefs. So for this special case, belief heterogeneity does not affect the real productive decisions. Detemple and Murthy (1994), Panageas (2005),

[1] See, for example, Zapatero (1998), Bhamra and Uppal (2014), Cvitanić, Jouini, Malamud and Napp (2011), Li (2007), Li (2013), Li and Muzere (2010), Yan (2008), and Loewenstein (2013)

and Baker, Hollifield and Osambela (2015) study heterogeneous beliefs in a production economy. However, in these papers, investors disagree about aggregate fundamentals. An important question then arises: what if investors agree on the aggregate fundamentals but disagree about events which do not affect fundamentals.

In this paper, we build a tractable model with investors who disagree about productivity growth or events which might not affect productivity at all, i.e. "sunspots" or some government policy which does not have any effect on real productivity growth. The model includes costs of capital adjustment so that as special subcases we can look at production technologies for which adjustment to the capital stock is irreversible, costless, or impossible.[2]

Agents are assumed to have the same coefficient of relative risk aversion and possibly heterogeneous time discount parameters. The productive technology is a linear production technology as in the Cox, Ingersoll and Ross (1985) with linear adjustment costs. We examine how heterogeneity of beliefs affect productive decisions and equilibrium asset prices.

We examine two forms of the financial market, one in which only trade in real risks is allowed and one in which trade in both types of risks are allowed. Arguably, some financial activity seems to involve trade in risks which do not directly affect real productivity. For example gold is generally considered a store of value but does not play much of a role as a productive input. A neutral monetary policy in which speculators disagree about the rate of money supply growth as in Xiong and Yan (2010) might also be a source of speculation in risks of this sort. More extreme examples might be speculation on tulip bulb prices or sports betting. While the issue of trade in extraneous risk has been analyzed in Basak (2000) and Xiong and Yan (2010), these have been in a pure exchange setting. How trade in these risks affect real productive decisions has yet to be considered.

Our model shows belief heterogeneity does affect real productive decisions, even when investors agree on productivity growth but disagree on extraneous events. Belief heterogeneity generally raises investment in productive technology when speculators have a coefficient of relative risk aver-

[2] In addition, the last two cases provide two polar extremes for which we can compare the current literature dealing with fixed aggregate consumption to a model in which productive activity responds to changes in the discount rates.

sion less than one and lowers investment in the productive technology when speculators have a coefficient of relative risk aversion greater than one. Trade in extraneous risk further reinforces this effect. This effect is driven by the response of speculators optimal consumption/investment plans to their perceived future investment opportunity set. When the coefficient of relative risk aversion is greater than one, speculators generally increase current consumption when they perceive better future investment opportunities. Therefore, on aggregate, investment in the productive technology will decrease. On the other hand, when the coefficient of relative risk aversion is less than one, the effects are reversed, that is financial market activity will increase aggregate investment in the productive technology. For example, our model shows that when the coefficient of relative risk aversion is greater than one, even though each investor would choose to invest in the productive technology in autarky, the presence of the financial market can imply that the equilibrium production choice will involve depleting capital to consume even in the presence of adjustment costs. Trade in extraneous risk makes this more pronounced. In this sense, the financial market can "compete" with the productive sector. However, when the coefficient of relative risk aversion is less than one, the effects are reversed and trade in the financial market can raise investment above level which each speculator would choose if they controlled the firm and there was no trade.

One of the surprising findings of our analysis is that these effects on output can be quite large even when one type of speculators dominates the wealth distribution. While speculators who have different beliefs do not have much wealth, their demand for consumption in states the more wealthy speculators view as more or less unlikely affect state prices. Unlike the pure exchange economy, where quantities are fixed, optimal production depends on state prices. For example, the coefficient of relative risk aversion is greater than one, lower prices of future consumption will tend to raise demand for current consumption. Fixing quantities, this implies the price of current consumption must go up in order to clear markets. However, when quantities adjust, production will respond to provide more current consumption.

Our specification of the adjustment costs implies there can be regions of capital depletion, no adjustment, and capital accumulation. In the regions of capital accumulation and capital depletion, we show the optimal consumption is a weighted average of the (possibly infeasible) optimal linear scale speculators would choose in those regions in autarky plus an adjustment term. This adjustment term is zero when the only source of heterogeneity is in time discount parameters. When investors have heterogeneous beliefs, the adjustment term is strictly positive when the coefficient of relative risk aversion is greater than one and strictly negative when the coefficient of relative risk aversion is less than one. When speculators agree on fundamental risk but disagree about extraneous risks, they would choose the same linear scale in each region. In this case, the effect of disagreement is unambiguous.

The behavior of asset prices is dramatically different in the no-adjustment region than in the other regions. In the no-adjustment region, asset prices look like those in a pure exchange economy and productive decisions do not respond to changes in these prices. However, the behavior of asset prices in the capital accumulation and depletion regions differs because the productive decisions respond to changes in financial markets. In particular, when trade in extraneous risk is allowed, in the accumulation and depletion regions, the value of aggregate production only reflects fundamental risk while in the no-adjustment region this value can reflect extraneous risks. The riskless rate in the capital accumulation and depletion regions is a weighted average of the riskless rates that would prevail in that region for each speculator in autarky when consumption is given by the choice of optimal linear scale. In the no-adjustment region, the riskless rate is a weighted average of the riskless rates for pure exchange economies populated by homogeneous investors plus an adjustment term. Similarly, we show that in the no-adjustment region, the output-price ratio is a weighted average of the output-price ratios in pure exchange economies with homogeneous investors plus an adjustment term. In both cases, the adjustment term is present only when there is disagreement and has an unambiguous sign depending on whether the coefficient of relative risk aversion is greater or less than one. In the capital accumulation and depletion regions the output-price ratio is trivially a weighted average of the output-price ratios in those regions for each speculator in autarky.

When we consider irreversible investments, when the coefficient of relative risk aversion is less than one, we find that even though no individual would want to accumulate capital in autarky, in equilibrium, the firm optimally can choose to accumulate capital despite the irreversibility. Trade in extraneous risk makes this possibility more likely. This finding provides new insight into how financial market activity might be associated with the recent boom and bust in the construction industry.[3] Because investors see attractive future investment opportunities, they choose to consume less and save more. Fixing aggregate supplies, discount rates go down the and price-output ratio goes up until at some point capital accumulation for future consumption becomes more attractive. This suggests an important feedback effect between financial market activity—especially purely speculative activity—and the real economy.

The paper is organized as follows. Section 2 of the paper describes the basic elements of our model. Section 3 characterizes the optimal production and asset prices in equilibrium. Section 4 provides comparative statics and illustrations of how speculative activity can affect the real economy, Section 5 applies our model to explain the observed boom-and-bust cycles observed in the US housing market, and Section 6 concludes. All proofs are in the appendix.

2 The Model

We consider a model of two groups of investors $i = 1, 2$. Each group has a probability space $(\Omega, \mathcal{F}, P^i)$ which will be described later. Heterogeneous beliefs are captured by the fact that the probability measure P^i is different across these groups of investors. We also allow for a heterogeneous time preference parameter ρ_i. Investors in our model have preferences over consumption plans given by

$$E^i \left[\int_0^\infty e^{-\rho_i t} \frac{c_{i,t}^{1-\gamma}}{1-\gamma} dt \right], \tag{2.1}$$

[3] Cheng, Raina and Xiong (2014) find that beliefs play a major role in the recent housing market boom and bust. Glaeser (2013) and Burnside, Eichenbaum and Rebelo (2015) use heterogeneous beliefs to explain the booms and busts in housing markets. The boom-and-bust cycles are also widely observed in other markets, for example the so-called "tech bubble" of stock market in the late 1990s and commodity price boom/bust in 2008, which is also attributed to heterogeneous beliefs by Singleton (2014).

where γ is the coefficient of relative risk aversion. We assume $0 < \gamma < 1$ or $\gamma > 1$.

Uncertainty in our model is generated by two independent Brownian motions B_t and $\hat{B}_t$. For concreteness we assume that these Brownian motions are defined on $(\Omega, \mathcal{F}, P^1)$ and we take the filtration to be the filtration generated these Brownian motions augmented by the P^1 null sets. Given capital stock K_t, the productive technology generates a continuous stream of consumption good at an annual rate of

$$\alpha A_t K_t, \tag{2.2}$$

where $\alpha > 0$ is a constant, and A_t, a conversion factor between capital and consumption good, is a geometric Brownian motion which follows (under agent 1's beliefs)

$$dA_t = \mu A_t dt + \sigma A_t dB_t. \tag{2.3}$$

Beliefs for agent 2 are described by the process N, $E^2[x] = E^1[N_t x]$ for any $x \in \mathcal{F}_t$ for any finite t, which evolves as

$$dN_t = -\beta N_t d\tilde{B}_t, \tag{2.4}$$

where $\tilde{B}_t = \eta B_t + \sqrt{1-\eta^2}\hat{B}_t$ and this implies $d\langle B, \tilde{B}\rangle = \eta dt$. Then under agent 2's probability measure

$$dA_t = (\mu - \eta\beta\sigma)A_t dt + \sigma A_t dB_t^2, \tag{2.5}$$

where $B_t^2 = B_t + \eta\beta t$ and $\hat{B}_t^2 = \hat{B}_t + \sqrt{1-\eta^2}\beta t$ are independent Brownian motions under agent 2's probability measure. If $\eta = 1$ they disagree only about the drift of productivity. If $\eta = 0$, they disagree on extraneous, or non-fundamental, events but agree on the drift of productivity.[4] For intermediate cases they disagree about both.

[4]We will see that that when $\eta = 0$ and $\rho_1 = \rho_2$, and there are no securities traded on this risk then the optimal production plan will be the same for each agent. Introducing securities which allow trade on the extraneous risk will change this however.

The aggregate budget constraint is[5]

$$I_t + c_t \le \alpha A_t K_t, \tag{2.6}$$

where I_t, measured in the consumption good, is the amount reinvested in capital stock and c_t is the consumption dividend paid to investors. In order to reinvest this consumption good into capital stock, an linear adjustment cost must be paid. In addition, we make an assumption that the technology to convert consumption into capital or capital into consumption is linear and proportional to $\frac{1}{A_t}$. Therefore, the capital stock evolves according to

$$dK_t = \left(-\delta K_t + \frac{1}{A_t} \left\{ I_t - a I_t 1_{\{I_t > 0\}} + b I_t 1_{\{I_t < 0\}} \right\} \right) dt, \tag{2.7}$$

where $\delta > 0$ is the capital depreciation rate, $a > 0$ and $b > 0$ are marginal adjustment costs for investment and divestment, respectively. The asymmetric marginal adjustment costs play an important role in our model, they also help to capture realistic features of adjustment costs for example, irreversible investment. In this case, our assumptions on the conversion of consumption into capital implies when A_t is low, then reinvested consumption goods can produce larger amounts of capital, while if A_t is high, then reinvested consumption goods produce less capital goods.

Since $I_t = \alpha A_t K_t - c_t$, we have

$$\begin{aligned} dA_t K_t &= A_t dK_t + K_t dA_t \\ &= A_t \left(-\delta K_t + \frac{1}{A_t} \left\{ I_t - a I_t 1_{\{I_t > 0\}} + b I_t 1_{\{I_t < 0\}} \right\} \right) dt + K_t \left(A_t \mu dt + A_t \sigma dB_t \right) \\ &= \left[A_t K_t \left(\mu - \delta \right) + \left(\alpha A_t K_t - c_t \right) \left(1 - a 1_{\{\alpha A_t K_t > c_t\}} + b 1_{\{\alpha A_t K_t < c_t\}} \right) \right] dt + A_t K_t \sigma \, dB_t. \end{aligned} \tag{2.8}$$

[5] This allows for free disposal; at an optimum the budget constraint will hold with equality so our further derivations will assume this holds with equality. Under the assumption of free disposal, the production set is convex.

2.1 Financial Markets

Individual i is endowed with θ_i shares in the firm. We assume $\theta_1 + \theta_2 = 1$. They can also potentially trade two financial securities in zero net supply. There is also locally riskless borrowing and lending in zero net supply. Prices in our model can be expressed in terms of consumption or in terms of capital. Using consumption as a numeraire and letting r_t be the risk-free rate, one unit of consumption invested in the locally riskless asset will evolve according to the equation

$$dR_t = r_t R_t dt + R_t dL_t, \tag{2.9}$$

where L_t in a continuous adapted finite variation process, which will play a role in equilibrium pricing as we explain later. A dollar invested in the financial assets evolves according to the equation

$$dF_t^j = \mu_{Ft}^j F_t^j dt + F_t^j dL_t + F_t^j \boldsymbol{\sigma}_F^j d\mathbf{Z}_t \tag{2.10}$$

for $j = 1, 2$, where

$$\mathbf{Z}_t = \begin{bmatrix} B_t \\ \hat{B}_t \end{bmatrix} \tag{2.11}$$

and

$$\boldsymbol{\sigma}_F^1 = \begin{bmatrix} \sigma_{1F}^1 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_F^2 = \begin{bmatrix} 0 & \sigma_{2F}^2 \end{bmatrix}. \tag{2.12}$$

Let

$$\boldsymbol{\mu}_{Ft} = \begin{bmatrix} \mu_{Ft}^1 \\ \mu_{Ft}^2 \end{bmatrix}, \quad \boldsymbol{\sigma}_F = \begin{bmatrix} \boldsymbol{\sigma}_F^1 \\ \boldsymbol{\sigma}_F^2 \end{bmatrix}. \tag{2.13}$$

We will examine how allowing trade in extraneous risk affects the economy. In this case we define state prices by letting $\boldsymbol{\kappa}_t = \boldsymbol{\sigma}_F^{-1}(\boldsymbol{\mu}_{Ft} - r_t\mathbf{1})$ and

$$\xi_t = \exp\left(\int_0^t \left(-r_s - \frac{1}{2}||\boldsymbol{\kappa}_s||^2\right) ds - L_t - \int_0^t \boldsymbol{\kappa}_s^\top d\mathbf{Z}_s\right). \tag{2.14}$$

As an alternative we will examine when trade in the extraneous risks is not possible. In this case we define state prices by letting $\kappa_t = \frac{\mu^1_{Ft} - r_t}{\sigma^1_{1F}}$ and

$$\xi_t = \exp\left(\int_0^t \left(-r_s - \frac{1}{2}\kappa_s^2\right) ds - L_t - \int_0^t \kappa_s dB_s\right). \tag{2.15}$$

We assume investors can pledge shares of the firm as collateral for their financial transactions[6]. Investors must maintain solvency, that is the value of their financial losses cannot exceed the value of their shares in the firm (valued in the consumption numeraire),

$$W_t^i \geq -\theta_i S_t,$$

where S_t is the value of the aggregate dividend paid to the investors, that is

$$S_t = \frac{1}{\xi_t} E^1\left[\int_t^\infty \xi_s c_s ds | \mathcal{F}_t\right]. \tag{2.16}$$

Investors trade the two financial securities and the riskless asset, receive the dividend from the firm and decide how much to consume. The wealth constraint for each investor can be written as

$$dW_t^i = \left(r_t W_t^i + \boldsymbol{\phi}_t^i(\boldsymbol{\mu}_{Ft} - r_t \mathbf{1}) + \theta_i c_t - c_{i,t}\right) dt + W_t^i dL_t + \boldsymbol{\phi}_t^i \boldsymbol{\sigma}_F \, d\mathbf{Z}_t, \tag{2.17}$$

where $\boldsymbol{\phi}_t^i = [\phi^i_{1,t} \; \phi^i_{2,t}]$ is the value invested in each financial security and $W_0^i = 0$.

Simple calculations give

$$\xi_t(W_t^i + \theta^i S_t) + \int_0^t \xi_s c_{i,s} ds \tag{2.18}$$

is a nonegative local martingale therefore

$$E^1\left[\int_t^\infty \xi_s c_{i,s} ds | \mathcal{F}_t\right] \leq \xi_t(W_t^i + \theta_i S_t). \tag{2.19}$$

[6] Allowing investors to trade their shares in the financial market will not change the equilibrium because the shares are a redundant asset. In this case we would use a nonnegative wealth constraint.

The investors problems can now be stated. Our first choice problem allows the investors to trade in both financial securities.

Choice Problem 2.1 (Choice problem with trade in real and non-fundamental risks)**.** *Choose consumption* $c_{i,t}$ *and trading strategies,* $\boldsymbol{\phi}_t^i$ *to maximize*

$$E^i\left[\int_0^\infty e^{-\rho_i t}\frac{c_{i,t}^{1-\gamma}}{1-\gamma}dt\right]$$

subject to the dynamic wealth constraint (2.17) *with the initial condition* $W_0^i = 0$*, and the solvency condition*

$$W_t^i \geq -\theta_i S_t. \tag{2.20}$$

Alternatively we examine the case where only the first financial security is traded so there is no trade in the non-fundamental risk $\hat{B}$.

Choice Problem 2.2 (Choice problem with trade in only real risk)**.** *Choose consumption* $c_{i,t}$ *and a trading strategy,* $\phi_{1,t}^i$ *to maximize*

$$E^i\left[\int_0^\infty e^{-\rho_i t}\frac{c_{i,t}^{1-\gamma}}{1-\gamma}dt\right]$$

subject to the dynamic wealth constraint

$$dW_t^i = \left(r_t W_t^i + \phi_{1,t}^i(\mu_{Ft}^1 - r_t) + \theta_i c_t - c_{i,t}\right)dt + W_t^i dL_t + \phi_{1,t}^i \sigma_{1,F}^1 dB_t \tag{2.21}$$

with the initial condition $W_0^i = 0$*, and the solvency condition*

$$W_t^i \geq -\theta_i S_t. \tag{2.22}$$

We can find the solution to the investors' optimization problems by writing a static maximization problem with a single linear budget constraint as in Cox and Huang (1989). The next proposition summarizes the optimal solutions for the investors for a given production plan.

Proposition 2.1. *1. The optimal solution to Problem 2.1 is given by*

$$c_{1,t} = \left(\chi_1 e^{\rho_1 t}\xi_t\right)^{-\frac{1}{\gamma}} \quad c_{2,t} = \left(\chi_2 e^{\rho_2 t}\frac{\xi_t}{N_t}\right)^{-\frac{1}{\gamma}}, \tag{2.23}$$

where ξ_t defined in Equation (2.14) and χ_i is the solution to

$$E^1\left[\int_0^\infty \xi_t c_{i,t} dt\right] = \theta_i S_0. \tag{2.24}$$

2. Let $\mathcal{F}_t^B$ be the filtration generated by the real risk B_t, and suppose the production choice $c_t \in \mathcal{F}_t^B$ and the state price density $\xi_t \in \mathcal{F}_t^B$. The optimal solution to Problem 2.2 is given by

$$c_{1,t} = \left(\chi_1 e^{\rho_1 t}\xi_t\right)^{-\frac{1}{\gamma}} \quad c_{2,t} = \left(\chi_2 e^{\rho_2 t}\frac{\xi_t}{E^1[N_t|\mathcal{F}_t^B]}\right)^{-\frac{1}{\gamma}}, \tag{2.25}$$

where ξ_t defined in Equation (2.15) and χ_i is the solution to

$$E^1\left[\int_0^\infty \xi_t c_{i,t} dt\right] = \theta_i S_0. \tag{2.26}$$

Proposition 2.1 gives important insight into the equilibrium with and without trade in extraneous risk. When there is trade in extraneous risk, then we have a standard complete market optimization problem. When there is no trade in the extraneous risk, while markets are incomplete, the resulting optimization problem can be thought of as one where a fictitious market for trade in the extraneous risk is introduced to complete the market. When the optimal production plan does not depend on extraneous risk, $\xi_t \in \mathcal{F}^B$, and the asset for extraneous risk has no risk premium, then the investor will not trade the fictitious asset so the complete market solution with the fictitious market is in fact the solution for the incomplete market problem[7]. When investors

[7]See He and Pearson (1991) or Karatzas, Lehoczky, Shreve and Xu (1991) for formal treatments of solving incomplete market consumption investment problems using this intuition. In our setting, the problem is much

disagree about extraneous risk it is impossible to introduce this second security with zero risk premium in both investors' probability measures so an equilibrium with the second security must reflect disagreement about the extraneous risk.

2.2 Equilibrium

In this section we provide definitions for equilibrium. Our definitions are straightforward. Given state prices, investors choose an optimal consumption and financial market trading plans, the production plan maximizes the value of the output, and given these choices markets clear.

Definition 2.1 (Equilibrium with trade in real and non-fundamental risks)**.** *An equilibrium with trade in real and non-fundamental risks consists of an adapted stochastic process ξ_t as in Equation* (2.14)*, feasible consumption and trading strategies $c_{i,t}$, $\boldsymbol{\phi}_t^i$, and feasible production output c_t such that*

1. *Given ξ_t, the consumption processes are optimal solutions to Problem 2.1.*

2. *The consumption market clears: $c_{1,t} + c_{2,t} = c_t$.*

3. *The asset markets clear: $\boldsymbol{\phi}_t^1 + \boldsymbol{\phi}_t^2 = \mathbf{0}$, $W_t^1 + W_t^2 = 0$.*

4. *Production c_t is chosen to maximize the value of the firm in Equation* (2.27)*.*

Definition 2.2 (Equilibrium with trade in only real risk)**.** *An equilibrium with trade in only real risk consists of an adapted stochastic process ξ_t as in Equation* (2.15)*, feasible consumption and trading strategies $c_{i,t}$, ϕ_{1t}^i, and feasible production output c_t such that*

1. *Given ξ_t, the consumption processes are optimal solutions to Problem 2.2.*

2. *The consumption market clears: $c_{1,t} + c_{2,t} = c_t$.*

3. *The asset markets clear: $\phi_t^1 + \phi_t^2 = 0$, $W_t^1 + W_t^2 = 0$.*

simpler so we do not need to employ these techniques.

4. Production c_t is chosen to maximize the value of the firm in Equation (2.27)*.*

Even though investors have different beliefs about the probability of different events, they agree on prices for payoffs contingent on these events. Therefore, given prices, they agree on the optimal production policy for the firm regardless of whether there is trade in the non-fundamental risk. To see this, recall Proposition 2.1 indicates when the extraneous risk is not traded and $\xi_t \in \mathcal{F}_t^B$ then investors will choose consumption which does not depend on the extraneous risk. Market clearing then implies that production cannot depend on extraneous risk as well. But this then implies $\xi_t \in \mathcal{F}_t^B$. This gives the following proposition.

Proposition 2.2. *Given prices, in equilibrium both investors agree on the optimal production policy for the firm if trade is allowed in the second financial security or if trade is not allowed in the second financial security. The optimal production policy for the firm is to maximize*

$$E^1 \left[\int_0^\infty \xi_t c_t \, dt \right], \tag{2.27}$$

where ξ_t is given by Equation (2.14) *when both financial securities are traded and ξ_t is given by Equation* (2.15) *when only the first financial security is traded.*

Importantly, Proposition 2.2 says that when extraneous risk is not traded, then investors will agree that the optimal production policy should not depend on extraneous risk. Given that the firm maximizes the expression in Equation (2.27) where ξ is given by Equation (2.15), then the aggregate output will not depend on extraneous risk. Given this, investors optimal consumption will not depend on extraneous risk, and given this, investors will agree that the optimal production should not depend on extraneous risk.

3 Equilibrium Production

A significant simplification of the problem is to run the production process to maximize a representative agent's utility of the form

$$E^1\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda\frac{c_{1,t}^{1-\gamma}}{1-\gamma}+(1-\lambda)e^{(\rho_1-\rho_2)t}N_t\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right)dt\right] \tag{3.28}$$

for some $\lambda \in [0,1]$. It is convenient to define the state variable $M_t = e^{(\rho_1-\rho_2)t}N_t$. The dynamics of the state variable are given by

$$dM_t = (\rho_1-\rho_2)M_t dt - \beta M_t d\tilde{B}_t. \tag{3.29}$$

We can then write the planner's problem as follows.

Choice Problem 3.1. *Choose feasible production output c_t and an allocation $c_{1,t}$, $c_{2,t}$ with $c_{1,t} + c_{2,t} = c_t$ to maximize*

$$E^1\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda\frac{c_{1,t}^{1-\gamma}}{1-\gamma}+(1-\lambda)M_t\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right)dt\right] \tag{3.30}$$

subject to the dynamics of M (3.29) and

$$dA_tK_t = \left[A_tK_t(\mu-\delta)+(\alpha A_tK_t-c_t)\left(1-a1_{\{\alpha A_tK_t>c_t\}}+b1_{\{\alpha A_tK_t<c_t\}}\right)\right]dt + A_tK_t\sigma dB_t. \tag{3.31}$$

Proposition 3.1. *Equilibrium production and consumption choices when trade is allowed in both real and non-fundamental risks can be equivalently described by the solution to the planner problem 3.1 for $\lambda = \frac{\frac{1}{\chi_1}}{\frac{1}{\chi_1}+\frac{1}{\chi_2}}$ where χ_1 and χ_2 are as described in Proposition 2.1.*

Alternatively, we can examine the case where individuals are not allowed to trade in both risks. Define $\mathcal{F}_t^B$ to be the filtration generated by real risk B_t. In this case we have the following result.

Proposition 3.2. *Equilibrium production and consumption choices when trade is allowed in only real risk can be equivalently described by the solution to the planner problem 3.1 with the additional constraints* $c_{1,t} \in \mathcal{F}_t^B$ *and* $c_{2,t} \in \mathcal{F}_t^B$ *where* $\lambda = \frac{\frac{1}{\chi_1}}{\frac{1}{\chi_1}+\frac{1}{\chi_2}}$ *where* χ_1 *and* χ_2 *are as described in Proposition 2.1.*

3.1 Model Solution

To characterize the equilibrium production we now solve Problem 3.1 for the case when both risks are traded in financial markets and for the case where only real risk is traded in the financial market. We define the value function given the initial conditions of A, K, and M and the dynamics in equations (3.31) and (3.29) as

$$V(AK, M) = \sup_{\{c_{1,t}, c_{2,t} | c_{1,t}+c_{2,t}=c_t\}} E^1\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda \frac{c_{1,t}^{1-\gamma}}{1-\gamma} + (1-\lambda)M_t \frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right) dt\right]. \tag{3.32}$$

According to Proposition 3.1 this problem then characterizes the equilibrium production choice and allocation in the economy where both real and non-fundamental risks are traded.

To characterize the value function in the economy where only the real risk is traded, we define the value function similarly but now according to Proposition 3.2 we replace the state variable M with the state variable $E^1[M|\mathcal{F}_t^B]$. For notational convenience we can continue to refer to the state variable M but change the dynamics to

$$dM_t = (\rho_1 - \rho_2)M_t dt - \eta\beta M_t dB_t, \tag{3.33}$$

which now reflects the dynamics of $E^1[M|\mathcal{F}_t^B]$.

We begin our analysis by recording the optimal production for each investor if they own the firm and there is no trade in financial markets. It is useful to introduce the following notation

$$\Gamma_1 = \rho_1 - (1-\gamma)\left(\mu - \delta - \frac{\gamma\sigma^2}{2}\right), \quad \Gamma_2 = \rho_2 - (1-\gamma)\left(\mu - \delta - \eta\beta\sigma - \frac{\gamma\sigma^2}{2}\right), \tag{3.34}$$

and for $i = 1, 2$,

$$\bar{c}_{ia} = \frac{\Gamma_i - (1-\gamma)\alpha(1-a)}{(1-a)\gamma}, \quad \bar{c}_{ib} = \frac{\Gamma_i - (1-\gamma)\alpha(1+b)}{(1+b)\gamma}. \tag{3.35}$$

We assume $\Gamma_i > 0$ for $i = 1, 2$. This condition is sufficient for the existence of autarky equilibria.

Proposition 3.3. *Assume $\Gamma_i > 0$ for all $i = 1, 2$. Then, at autarky equilibrium,*

$$V(AK, M) = \frac{(AK)^{1-\gamma}}{1-\gamma} \times \begin{cases} h_1, & \lambda = 1 \\ Mh_2, & \lambda = 0, \end{cases} \tag{3.36}$$

where[8]

$$h_i = \begin{cases} \frac{\bar{c}_{ia}^{-\gamma}}{1-a}, & (1-\gamma)(1-a) < \frac{\Gamma_i}{\alpha} < 1-a \\ \frac{\alpha^{1-\gamma}}{\Gamma_i}, & 1-a \leq \frac{\Gamma_i}{\alpha} \leq 1+b \quad \text{or} \quad \frac{\Gamma_i}{\alpha} \leq (1-\gamma)(1-a) \\ \frac{\bar{c}_{ib}^{-\gamma}}{1+b}, & \frac{\Gamma_i}{\alpha} > 1+b, \end{cases} \tag{3.37}$$

for capital accumulation, no adjustment, and capital depletion, respectively. The corresponding consumption to AK ratios are $\bar{c}_{ia}$, α, and $\bar{c}_{ib}$.

Proposition 3.3 tells us the solution when there is only one type of investor present in the economy. In this case, the optimal ratio of consumption to AK is constant and the optimal choice is obtained by maximizing the Bellman equation over each of the three possible regions. In this case, an investor has three choices of production plan, capital accumulation, no adjustment, and capital depletion. The maximizing choice for each region must also be feasible. For example, in the capital accumulation region $\bar{c}_{1a}$ represents the optimal ratio of consumption to AK; if it is greater than α, then capital accumulation is feasible for investor 1. Similarly, if $\bar{c}_{1b}$ is smaller than α then capital depletion is not feasible. By solving this in all three regions, investors then choose the production (consumption) plan that is feasible because the conditions for feasibility are mutually exclusive. These polar solutions provide the key boundary conditions for solving the general case. Our next proposition provides some useful bounds and properties of the value function when there

[8] The condition $(1-\gamma)(1-a) < \frac{\Gamma_i}{\alpha}$ only applies to the case with $\gamma < 1$; it always satisfies for $\gamma > 1$.

is trade in the financial markets and both types of investor are present.

Proposition 3.4. *The value function $V(AK, M)$ has the form*

$$V(AK, M) = \frac{(AK)^{1-\gamma}}{1-\gamma} h(M).$$

The function $h(M)$ is strictly increasing in M, the function $h(M)/M$ is strictly decreasing in M, $\frac{h(M)}{1-\gamma}$ is convex, and

$$0 \leq \frac{Mh'(M)}{h(M)} \leq 1. \tag{3.38}$$

These properties of $h(M)$ in Proposition 3.4 are important to understand the equilibrium of the economy. Our next result characterizes the value function when both types of investors are present.

Proposition 3.5. *Suppose both financial securities are traded. If there is a C^2 function h which satisfies the conditions of Proposition 3.4 and solves*

$$\left(\frac{-\Gamma_1}{1-\gamma} + \alpha(1-a)\right) h(M) + \frac{\beta^2 M^2}{2(1-\gamma)} h''(M) + \left(\frac{\rho_1 - \rho_2}{1-\gamma} - \eta\beta\sigma\right) Mh'(M) \\ + \frac{\gamma}{1-\gamma}\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M)^{\frac{1}{\gamma}}\right]\left[(1-a)h(M)\right]^{1-\frac{1}{\gamma}} = 0 \tag{3.39}$$

in the capital accumulation region,

$$\left(\frac{-\Gamma_1}{1-\gamma} + \alpha(1+b)\right) h(M) + \frac{\beta^2 M^2}{2(1-\gamma)} h''(M) + \left(\frac{\rho_1 - \rho_2}{1-\gamma} - \eta\beta\sigma\right) Mh'(M) \\ + \frac{\gamma}{1-\gamma}\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M)^{\frac{1}{\gamma}}\right]\left[(1+b)h(M)\right]^{1-\frac{1}{\gamma}} = 0 \tag{3.40}$$

in the capital depletion region, and in the no-adjustment region

$$\frac{-\Gamma_1}{1-\gamma}h(M) + \frac{\beta^2 M^2}{2(1-\gamma)}h''(M) + \left(\frac{\rho_1 - \rho_2}{1-\gamma} - \eta\beta\sigma\right) M h'(M) + \frac{\alpha^{1-\gamma}}{1-\gamma}\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M)^{\frac{1}{\gamma}}\right]^{\gamma} = 0, \quad (3.41)$$

with the conditions

$$(1-a)h(M) \leq \alpha^{-\gamma}\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M)^{\frac{1}{\gamma}}\right]^{\gamma} \leq (1+b)h(M). \quad (3.42)$$

Boundary conditions are given by

$$\lim_{M\to 0} h(M) = \lambda h_1, \qquad \lim_{M\to\infty} \frac{h(M)}{M} = (1-\lambda)h_2, \quad (3.43)$$

where h_1 and h_2 are given in Proposition 3.3. Then $V(AK, M) = \frac{(AK)^{1-\gamma}}{1-\gamma}h(M)$.

Proposition 3.5 characterizes the optimal production plan when investors trade both financial assets. In contrast with the case where there is only one type of agent present, the optimal consumption to capital ratio can vary stochastically over time. A key implication of the model is that the nature of the optimal production plan also varies with M; it is possible that the economy switches from capital accumulation/depletion to the phase of no adjustment, in which all output is consumed and there is no aggregate investment or divestment. We also call the state of no adjustment the exchange phase. The economy in our model is a combination of pure production and pure exchange phases, which are endogenously determined in our model. The next proposition shows the quantitative effects of speculation on optimal production.

Proposition 3.6. *The aggregate consumption is given by*

$$c_t^* = c_{1,t}^* + c_{2,t}^* = A_t K_t \times \begin{cases} \left(\frac{\lambda}{(1-a)h(M_t)}\right)^{\frac{1}{\gamma}} + \left(\frac{(1-\lambda)M_t}{(1-a)h(M_t)}\right)^{\frac{1}{\gamma}} \\ \alpha \\ \left(\frac{\lambda}{(1+b)h(M_t)}\right)^{\frac{1}{\gamma}} + \left(\frac{(1-\lambda)M_t}{(1+b)h(M_t)}\right)^{\frac{1}{\gamma}}, \end{cases} \tag{3.44}$$

in the capital accumulation, no adjustment, and the capital depletion regions. Furthermore, the aggregate consumption also satisfies

$$c_t^* = A_t K_t \times \begin{cases} \left(1 - \frac{M_t h'(M_t)}{h(M_t)}\right) \bar{c}_{1a} + \frac{M_t h'(M_t)}{h(M_t)} \bar{c}_{2a} - \frac{\beta^2 M_t^2 h''(M_t)}{2\gamma(1-a)h(M_t)} \\ \left(1 - \frac{M_t h'(M_t)}{h(M_t)}\right) \bar{c}_{1b} + \frac{M_t h'(M_t)}{h(M_t)} \bar{c}_{2b} - \frac{\beta^2 M_t^2 h''(M_t)}{2\gamma(1+b)h(M_t)}, \end{cases} \tag{3.45}$$

in the capital accumulation and capital depletion regions, respectively.

Proposition 3.4 shows $h''(M)$ is negative (positive) when $\gamma < 1$ ($\gamma > 1$), and $0 \leq \frac{Mh'(M)}{h(M)} \leq 1$. Equation (3.45), in conjunction with the properties of $h(M)$, highlights the influence of disagreement on optimal production. If both investors would choose to accumulate capital in autarky and $\gamma < 1$ then in equilibrium the optimal production will accumulate capital. On the other hand, if both investors would choose to deplete capital in autarky and $\gamma > 1$ then in equilibrium the firm would deplete capital. In general, Proposition 3.6 shows that when $\gamma < 1$ heterogeneous beliefs tend to increase equilibrium investment while if $\gamma > 1$ equilibrium investment tends to decrease.

When $\beta = 0$, individuals have homogeneous beliefs but possibly different time preferences. In this case we see that the optimal consumption to AK ratio, $c_t^*/A_t K_t$, is a weighted average of $\bar{c}_{1a}$ and $\bar{c}_{2a}$ in the capital accumulation region. In this case, the optimal production must be smaller than α to be in the capital accumulation region, but also must be between $\bar{c}_{1a}$ and $\bar{c}_{2a}$. If $\bar{c}_{1a}$ and $\bar{c}_{2a}$ are both greater than α then the optimal production cannot involve capital accumulation. However, if $\beta \neq 0$, then optimal production can involve capital accumulation when $\gamma < 1$ since in this case Proposition 3.4 implies $h''(M) \geq 0$ so the additional term can lower the optimal consumption and raise investment.

A similar analysis applies when $\gamma > 1$. In this case Proposition 3.4 implies $h''(M) \leq 0$ so disagreement will cause optimal consumption to AK ratio to be higher than a weighted average of $\bar{c}_{1b}$ and $\bar{c}_{2b}$ in the capital depletion region. If $\bar{c}_{1b}$ and $\bar{c}_{2b}$ are both less than α, and $\beta = 0$, then optimal production cannot involve capital depletion. However, when $\beta \neq 0$, then optimal consumption can be higher and capital depletion can be part of the optimal production plan.

The next proposition outlines the solution for the function h when only real risks are traded. Trade in the non-fundamental risk either lowers the trough optimal consumption to AK ratio or raises the peak optimal consumption to AK ratio depending on the coefficient of relative risk aversion. Without trade in non-fundamental risks, the trough optimal consumption to AK ratio is higher than the economy where non-fundamental risks are traded when $0 < \gamma < 1$ and the peak optimal consumption to AK ratio is lower than the economy where non-fundamental risk is traded when $\gamma > 1$.

Proposition 3.7. *When only real risks are traded, if a function h satisfies the same conditions as in Proposition 3.5 except the term $\frac{\beta^2 M^2}{2(1-\gamma)} h''(M)$ is replaced by $\frac{\eta^2 \beta^2 M^2}{2(1-\gamma)} h''(M)$ in equations (3.39), (3.40), and (3.41), then the value function is $V(AK, M) = \frac{(AK)^{1-\gamma}}{1-\gamma} h(M)$. Optimal consumption is again given by (3.44) and when $0 < \gamma < 1$ ($\gamma > 1$) the maximal (minimal) optimal consumption to AK ratio is always higher (lower) than that in the economy when all risks are traded.*

To understand the effects of speculation over extraneous risk, we can insert the expression for optimal consumption in Equation (3.44) into Equations (3.39), and (3.40) only with the term $\frac{\beta^2 M^2}{2(1-\gamma)} h''(M)$ is replaced by $\frac{\eta^2 \beta^2 M^2}{2(1-\gamma)} h''(M)$. This gives Equation (3.45) with β^2 replaced by $\eta^2 \beta^2$. Since $|\eta| \leq 1$ this is suggestive of the fact that trading extraneous risk will tend to have a bigger effect on real activity. In particular, when $\eta = 0$ and $\beta \neq 0$, then investors agree on fundamental risk but disagree on extraneous risk. In this case, trading extraneous risk raises equilibrium consumption above the weighted average when $\gamma > 1$ and lowers equilibrium consumption below the weighted average when $\gamma < 1$. However, when extraneous risk is not traded, then aggregate consumption must be equal to the weighted average.

Propositions 3.5 and 3.7 indicate that the function h is the solution to a free boundary problem. In other words, we must find the boundaries between the accumulation, no-adjustment, and depletion regions. To solve this we need to impose smooth pasting conditions at the boundaries between the various regions to ensure the function h is C^2 . This is aided by the fact that the solution in the no-adjustment region takes a very explicit form[9]

$$h(M) = \left(C_1 - \frac{2\alpha^{1-\gamma}}{\beta^2(\phi_+ - \phi_-)} \int^M \frac{\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)x)^{\frac{1}{\gamma}}\right]^\gamma}{x^{\phi_+ + 1}} dx \right) M^{\phi_+} + \left(C_2 + \frac{2\alpha^{1-\gamma}}{\beta^2(\phi_+ - \phi_-)} \int^M \frac{\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)x)^{\frac{1}{\gamma}}\right]^\gamma}{x^{\phi_- + 1}} dx \right) M^{\phi_-} \tag{3.46}$$

for constants ϕ_+ and ϕ_- as defined in Equation (6.94) in the appendix.

Proposition 3.8. *The function $h(M)$ in the pure exchange region is given by Equation 3.46. Given C^2 candidate solutions outside of the pure exchange region, if C_1, C_2 and the boundaries are chosen so that the value and derivative match at the boundaries, the resulting function is C^2.*

In order to solve the free boundary problem, we employ a variation of the "shooting" method in which a boundary value problem is transformed into an initial value problem. The basic idea is that we know the boundary values for $\lim_{M\to 0} h(M)$ and $\lim_{M\to\infty} \frac{h(M)}{M}$. For concreteness, let us assume that when $M = 0$ we are in the capital accumulation region, in other words investor 1 in autarky would run the firm to accumulate capital. We start very near $M = 0$ and prescribe an initial value for $h'(M)$. We then numerically solve (3.39) with these initial conditions. We then travel along this solution until (3.42) holds with equality. At this point we solve for C_1 and C_2 in our expression for the value function in the no-adjustment region so that this expression matches the value of our numerical solution and so that the derivatives also match. We then travel along this solution until the one of the boundary conditions (3.42) are violated. This then gives us initial

[9] A brief proof is provided in the proof of Proposition 3.5. This expression also leads to a closed form expression (see Proposition 4.1) for the value of the aggregate endowment in a pure exchange economy. This, and much more, is analyzed in Lee, Li and Loewenstein (2015).

conditions at this point to solve either (3.39) or (3.40) depending on which boundary condition is violated. We then travel along this solution and continue in this manner until we reach a very large value of M. At this point we then compare the value of $\frac{h(M)}{M}$ for our constructed solution to the known value we are trying to match. We then repeat this procedure by varying the initial condition for $h'(M)$ until we satisfy the boundary condition for very large M. By construction, the smooth pasting and value matching conditions hold at the boundaries of the adjustment regions and the no-adjustment regions.

The nature of the optimal consumption helps determine where the optimal boundaries are located. From Proposition 3.6, we know that if both investors would choose to invest capital in autarky and $\gamma < 1$ then in equilibrium, the firm would invest capital. Similarly, if $\gamma > 1$ and both speculators would choose to deplete in autarky, then in equilibrium, the firm would always deplete. The cases which involve capital accumulation, no-adjustment, and capital depletion can arise when the optimal consumption to capital ratio in autarky is to accumulate capital for one speculator and to deplete capital in autarky for the other speculator. However, this is not necessary. Interestingly, when $\gamma < 1$ if both investors would choose to deplete capital, it is possible for the equilibrium production choice to involve no-adjustment and/or capital investment. In this case we look for 4 possible boundaries: the first boundary between the capital depletion and the no adjustment region, the first boundary between the no-adjustment region and the capital accumulation region, the second boundary between the capital accumulation and the capital adjustment region, and the second boundary between the no-adjustment region and the capital depletion region. Similarly, when $\gamma > 1$ if both investors would choose to deplete capital in autarky, the equilibrium production choice could involve no adjustment and/or capital accumulation, and again we would look for four possible boundaries.

3.2 Asset Prices

In equilibrium, optimality requires

$$c_{1,t}^{1-\gamma} = \frac{1}{\lambda}\xi_t. \tag{3.47}$$

While the value function $V(AK, M)$ is C^2 in M, consumption as a function of M is not differentiable at the boundary points where the economy switches between productive and pure exchange regions. This occurs because the adjustment cost function is not differentiable. Therefore, one of the interesting features of this model with linear adjustment costs is that the rate of interest may not exist. That is, the value of a one unit of consumption invested in riskless lending is given by

$$R_t = \exp\left(\int_0^t r_s ds + L_t\right), \tag{3.48}$$

where L_t is a measurable, adapted, continuous finite variation process which is singular with respect to Lesbesgue measure, meaning it increases or decreases only on a Lesbesgue measure zero set of points. The presence of L_t is due to non-smoothness of consumption in the state variable M at the boundaries of the no-adjustment region. The function L_t only increases or decreases at these points depending on whether $c_{1,t}(AK, M)$ is a concave or convex function of M. To see this, we use the generalized Ito formula for convex, not necessarily differentiable, functions (Karatzas and Shreve (1991) Theorem 3.7.1) to expand $c_{1,t}(AK, M)$ and match coefficients for the Ito expansion for ξ_t to solve for L_t, r_t, and κ_t.

The interest rate, when it exists, is the expected growth rate of marginal utility of consumption. In the adjustment regions, marginal utility of consumption is proportional to the marginal utility of capital $(AK)^{-\gamma}h(M)$. In the no-adjustment region, marginal utility of consumption is proportional to $(\alpha AK)^{-\gamma}\left[\lambda^{\frac{1}{\gamma}} + (1-\lambda)M)^{\frac{1}{\gamma}}\right]^{\gamma}$. While consumption is continuous in M, it is not differentiable at the boundaries of a no-exchange region. In the interior of the adjustment regions and the no-adjustment region, the expected growth rate is found using Ito's lemma. However, at the boundaries, consumption is not smooth in M and the process L_t accounts for this. Practically speaking, to measure the r_t and L_t processes one would need to observe a continuous path of the investment in the locally riskless asset; these processes cannot be identified with discrete data. However, the following proposition suggests investments in the locally riskless asset will have very different properties across the different phases of the economy.

Proposition 3.9. *When trade is allowed in both financial securities, the state price density (under P^1) is given by*

$$\xi_t = \begin{cases} e^{-\rho_1 t}(A_t K_t)^{-\gamma} h(M_t)(1-a)/(c^*_{1,0})^{-\gamma} \\ e^{-\rho_1 t}(\alpha A_t K_t)^{-\gamma}\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M_t)^{\frac{1}{\gamma}}\right]^{\gamma}/(c^*_{1,0})^{-\gamma} \\ e^{-\rho_1 t}(A_t K_t)^{-\gamma} h(M_t)(1+b)/(c^*_{1,0})^{-\gamma} \end{cases}$$

and the interest rate is given by

$$r_t = \begin{cases} \alpha(1-a) + \mu - \delta - \gamma\sigma^2 - \eta\beta\sigma\frac{M_t h'(M_t)}{h(M_t)} \\ \frac{\lambda^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M_t)^{\frac{1}{\gamma}}}\bar{r}_1 + \frac{((1-\lambda)M_t)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M_t)^{\frac{1}{\gamma}}}\bar{r}_2 + \frac{1}{2}\left(1-\frac{1}{\gamma}\right)\frac{(\lambda(1-\lambda)M_t)^{\frac{1}{\gamma}}}{\left(\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M_t)^{\frac{1}{\gamma}}\right)^2}\beta^2 \\ \alpha(1+b) + \mu - \delta - \gamma\sigma^2 - \eta\beta\sigma\frac{M_t h'(M_t)}{h(M_t)}, \end{cases} \tag{3.49}$$

for the regions of capital accumulation, no adjustment, and capital depletion, respectively, where

$$\begin{aligned} \bar{r}_1 &= \rho_1 + \gamma(\mu-\delta) - \frac{1}{2}\gamma(1+\gamma)\sigma^2, \\ \bar{r}_2 &= \rho_2 + \gamma(\mu-\delta-\eta\beta\sigma) - \frac{1}{2}\gamma(1+\gamma)\sigma^2. \end{aligned}$$

The interest rate does not exist at the transition points between regions of capital accumulation and no adjustment, and capital depletion and no adjustment.

When trade in only real risk is allowed the expressions above still apply with β^2 replaced by $\eta^2\beta^2$.

Proposition 3.9 indicates that asset prices behave differently in the accumulation and depletion phases than in the exchange phase. To better understand this, we first consider the behavior of the interest rate. For example, when investors have identical beliefs, the equilibrium interest rate in the accumulation phase is given by

$$\alpha(1-a)+\mu-\delta-\gamma\sigma^2, \quad \alpha(1-a)+\mu-\eta\beta\sigma-\delta-\gamma\sigma^2$$

if they both have investor 1's beliefs or if they both have investor 2's beliefs, respectively. Notice that these do not depend on the time discount rate. In equilibrium with investors with both types of beliefs, the interest rate in the accumulation phase is a weighted average these interest rates with the weights

$$1-\frac{Mh'(M)}{h(M)}, \quad \frac{Mh'(M)}{h(M)},$$

which are between 0 and 1 as discussed earlier.

In exchange phase, the interest rate when all investors have the same time preferences and beliefs as investor i is $\bar{r}_i$, given in Proposition 3.9. In contrast with the rates in the production regions, this does depend on the time preference. In equilibrium with both types of investors, the equilibrium interest rate is not simply weighted sum of all r_i, but it also includes the term with β^2 or $\eta^2\beta^2$, which is positive when $\gamma > 1$ and negative when $\gamma < 1$. This additional term arises because investors with $\gamma > 1$ prefer to consume earlier and invest less when future investment opportunities become more attractive, while investors with $\gamma < 1$ prefer to invest more and consume later when future investment opportunities become more attractive.

Much of the behavior of asset prices is traced to the fact that in the accumulation and depletion regions the marginal utility of consumption is proportional to the marginal utility of capital; in the exchange phase, this proportionality does not hold.

Proposition 3.10. *The value of aggregate consumption (or dividend) is given by*

$$S_t = \alpha A_t K_t \times \begin{cases} \frac{1}{(1-a)\alpha} \\ \frac{\alpha^{\gamma-1} h(M_t)}{\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M_t)^{\frac{1}{\gamma}}\right]^{\gamma}} \\ \frac{1}{(1+b)\alpha}, \end{cases} \tag{3.50}$$

for the regions of capital accumulation, no adjustment, and capital depletion, respectively. Furthermore, in the no-adjustment region

$$S_t = \alpha A_t K_t \times \left[\left(1 - \frac{M_t h'(M_t)}{h(M_t)}\right) \Gamma_1 + \frac{M_t h'(M_t)}{h(M_t)} \Gamma_2 - \frac{\beta^2}{2} \frac{M_t^2 h''(M_t)}{h(M_t)} \right]^{-1}, \tag{3.51}$$

where Γ_i*, given by* (3.34)*, is the output-price ratio in autarky in the case of no adjustment.*

Proposition 3.10 indicates that the output-price ratio, $\frac{\alpha A_t K_t}{S_t}$ is trivially a weighted average of the output-price ratios in an economy with homogeneous investors in the capital accumulation and depletion regions. In the no-adjustment region the output-price ratio is a weighted average of output-price ratios from a pure exchange economies with homogeneous investors plus an adjustment term. The adjustment term is non-zero when investors disagree and is strictly negative for $\gamma < 1$ and strictly positive for $\gamma > 1$. When $\gamma < 1$ ($\gamma > 1$), it is possible for the output-price ratio to be lower (higher) than the minimum output-price ratio in the single agent economies,$\min\{\Gamma_1, \Gamma_2\}$ ($\max\{\Gamma_1, \Gamma_2\}$), if the adjustment term for disagreement is large enough. The price-output ratio satisfies

$$\frac{S_t}{\alpha A_t K_t} > (<) \left[\left(1 - \frac{M_t h'(M_t)}{h(M_t)}\right) \Gamma_1 + \frac{M_t h'(M_t)}{h(M_t)} \Gamma_2 \right]^{-1}$$

for $\gamma < 1$ ($\gamma > 1$) in the no-adjustment region. When speculators have the same time discount rate and agree on the fundamental risk, the impact of speculation on asset price is unambiguous in the no-adjustment region; the price-output ratio is always higher (lower) than $\Gamma_1 = \Gamma_2$ when $\gamma < 1$ ($\gamma > 1$).

The price-output ratio is $1/(1-a)\alpha > 1/\alpha$ for capital accumulation region and $1/(1+b)\alpha < 1/\alpha$ for capital depletion region. However, this ratio becomes stochastic in the region of no adjustment, in which the economy becomes a pure exchange one. When the economy in the exchange phase, real investment does not respond to financial markets, so asset prices behave quite differently than in the accumulation and depletion regions. In particular, the stock price can be depend on the extraneous risk when the financial market trades both financial assets. For the accumulation/depletion phase, speculation does not directly impact this ratio, but does impact aggregate consumption and investment. The investment/divestment channel is optimally shutdown in the exchange phase, and thus heterogeneity between investors is fully reflected in asset prices and the interest rate.

When extraneous risk is traded, it affects the equilibrium through the dynamics of M_t. Since S_t does not depend on M_t in the production phase, the value of aggregate consumption does not depend on extraneous risk, however, it does depend on the extraneous risk in the exchange phase. Thus, the market value of aggregate consumption may have a lower correlation with fundamentals when investors have a large disagreement on extraneous risk.

4 Comparative Statics

We now explore the quantitative features of the model through numerical examples. In the numerical examples which follow, we consider two values of relative risk aversion; $3 = \gamma > 1$ and $0.3 = \gamma < 1$. For both of these choices, we fix the beliefs of investor 1 and let $\beta = -1$. This means that when $\eta = 1$ investor 2 estimates the cash flow growth to be one standard deviation of fundamental risk (σ) higher than investor 1. We will vary η to get different mixes of disagreement on fundamental and extraneous risk. When $\eta = 0$ both investors agree on the drift of the cash flows but have disagreement on extraneous risk. We examine the cases where η is positive which means that investor 1 is relatively pessimistic and investor 2 is relatively optimistic about the fundamental risk. For most of our analysis we assume the investors have the same time preference parameter $\rho_1 = \rho_2 = 0.05$. For the fundamental parameters, we choose $\alpha = 0.05$, $\mu - \delta = -0.02$, $\sigma = 0.03$ and

$\lambda = \frac{1}{2}$.

4.1 Aggregate Consumption and Investment when Both Financial Securities Are Traded

In this section, we assume both financial securities are traded and explore the effect of this on the optimal production in equilibrium. In Figure 1, we set $\gamma = 3$ and plot the ratio of aggregate consumption to AK versus the logarithm of M for different values of η. Consistent with Proposition 3.6 we see that heterogeneity of beliefs tends to raise aggregate consumption and thus decrease aggregate investment.

The top graph of Figure 1 displays the case where $\eta = 0$, that is investors agree on the fundamentals but disagree on extraneous risk. In this case, the limit as M approaches infinity and as M approaches zero are the same because in isolation (or when extraneous risk is not traded) each investor would choose the same production plan and consume approximately $0.036AK$. In this case, both investors would choose to accumulate capital since $0.036 < 0.05 = \alpha$. However, when both investors are present in equilibrium and aggregate risk is traded, aggregate consumption can be much higher than this level, and hence the aggregate investment is much lower. This indicates that speculation over extraneous risk can have large effects on the real side of the economy. It is also important to note that this effect is present even when one group of investors is relatively small; the aggregate consumption is above $0.04AK$ for $e^{-5} \leq M \leq e^{5}$.

When $\eta > 0$, investors disagree on the fundamental risk. Graphs 2 to 4 in Figure 1 plot the optimal consumption for different values of η. As η increases, investor 2 becomes more optimistic about the fundamental risk, thus, since $\gamma > 1$ in autarky he would choose a production plan which consumes more and invests less. In the second graph from the top, even though both investors still prefer capital accumulation, the aggregate consumption ratio is higher than $0.05 = \alpha$ for some values of M, thus two no-adjustment regions emerge to bridge the regions of capital accumulation and capital depletion. In the third graph of Figure 1, investor 2 is even more optimistic about the

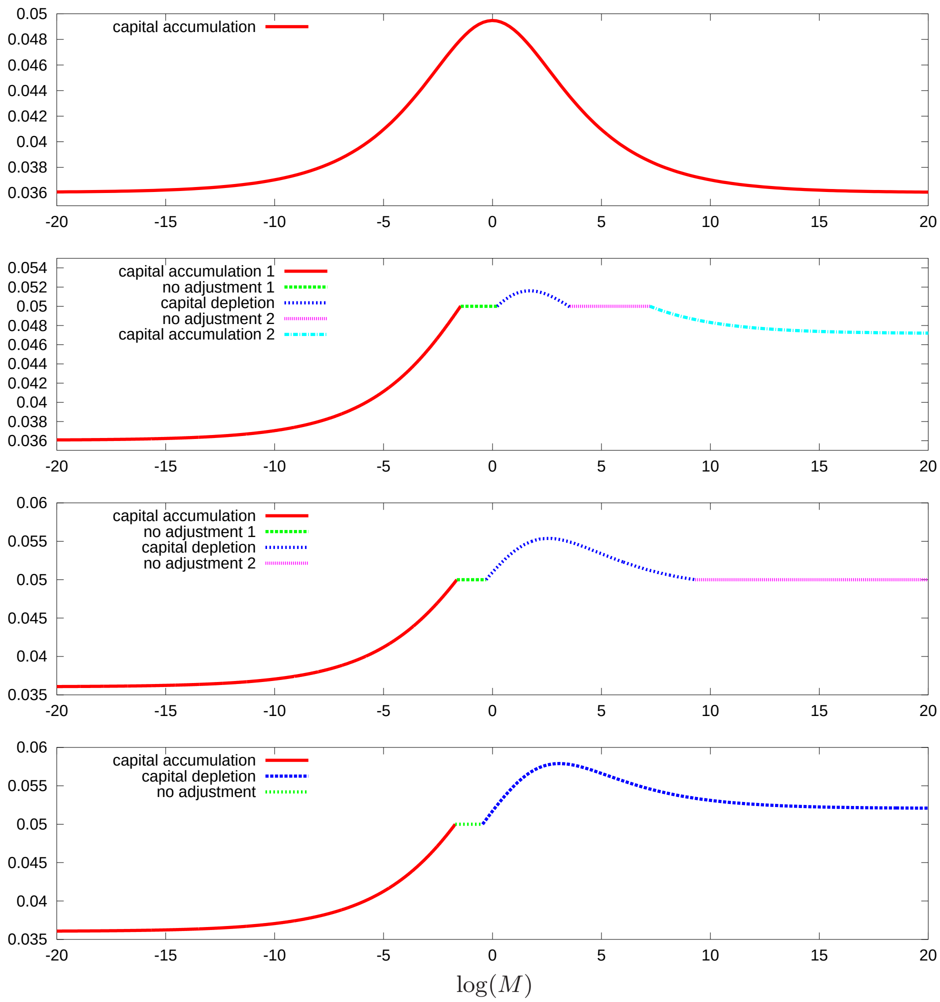

Figure 1: Aggregate Consumption to AK Ratio ($\gamma > 1$). The model parameters are set as follows: $\alpha = 0.05$, $\mu - \delta = -2\%$, $\sigma = 3\%$, $\rho_1 = \rho_2 = 5\%$, $\gamma = 3$, $\lambda = 0.5$, $\beta = -1$, $a = 0.1$, and $b = 0.2$. From top to bottom: $\eta = 0$, 0.5, 0.8, 1, respectively.

fundamental risk and he would choose a production plan with no adjustment to capital stock. In this case, the second no-adjustment region extends all the way to the boundary and the region of capital depletion also becomes larger. As investor 2 becomes more optimistic about the fundamental risk, he prefers capital depletion as indicated in the fourth graph in Figure 1, in which the second no-adjustment region vanishes and the magnitude of the depletion ratio is more pronounced.

We repeat the previous numerical exercises for the case of $\gamma < 1$ in Figure 2. A key difference is that the optimistic investors in autarky would run the firm to accumulate capital and consume a lower fraction of AK in general. According to Proposition 3.6 we should expect speculation to lower consumption, which is the case in Figure 2.

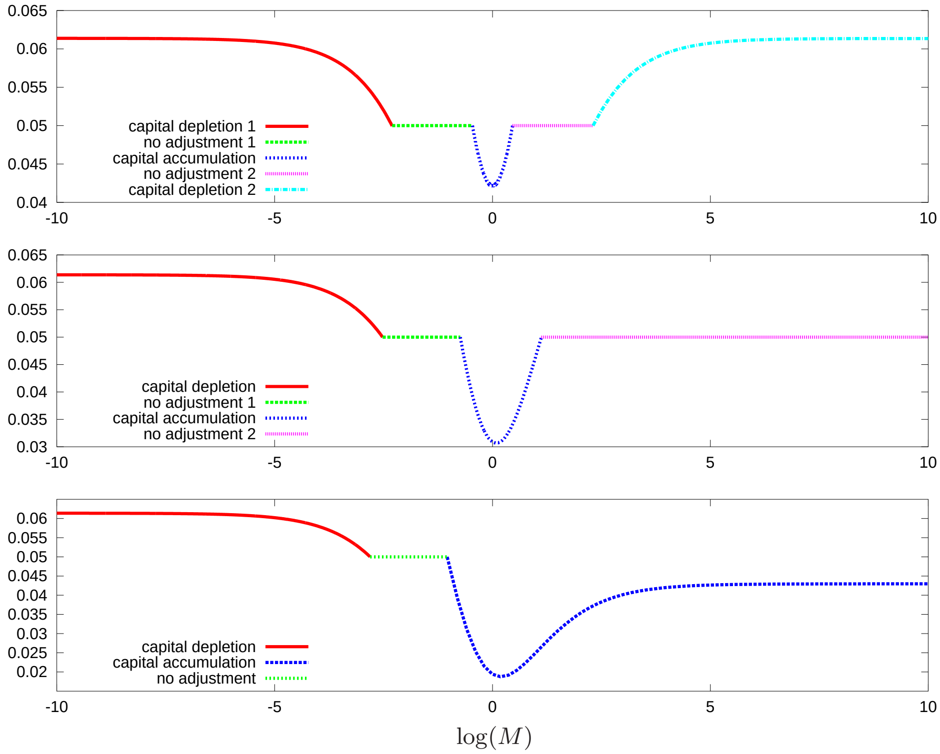

Figure 2: Aggregate Consumption to AK Ratio ($\gamma < 1$). The model parameters are set as follows: $\alpha = 0.05$, $\mu - \delta = -2\%$, $\sigma = 3\%$, $\rho_1 = \rho_2 = 5\%$, $\gamma = 0.3$, $\lambda = 0.5$, $\beta = -1$, $a = 0.1$, and $b = 0.2$. From top to bottom: $\eta = 0$, 0.5, 1, respectively.

When investors only disagree on extraneous risk, the production plan investors would choose in autarky (or in the case where extraneous risk is not traded) depletes capital and consumes at a higher rate than α as shown in the top graph of Figure 2. However, speculation changes the nature of the optimal consumption/investment policy. When both financial securities are traded, the optimal production plan accumulates capital when the future investment opportunity set is most attractive, that is when M near to 1.

As investor 2 becomes more optimistic about the fundamental risk, he prefers less current consumption (ratio), thus his investment in autarky goes from capital depletion to no adjustment to capital accumulation as η increases as shown in the second and third graphs in Figure 2. However, in equilibrium, when both financial securities are traded, consistent with Proposition 3.6 we see that the optimal production accumulates even more capital and consumption is lower.

In the regions of no adjustment, all output is consumed, thus, the economy is similar to a pure exchange economy, although this phase appears endogenously. As shown in the figures, adjustment costs make this production economy a combination of pure exchange, production with investment in capital, and production with divestment of capital. The switches among those different types of economy creates interesting dynamics of asset prices, which we examine next.

4.2 Asset Prices

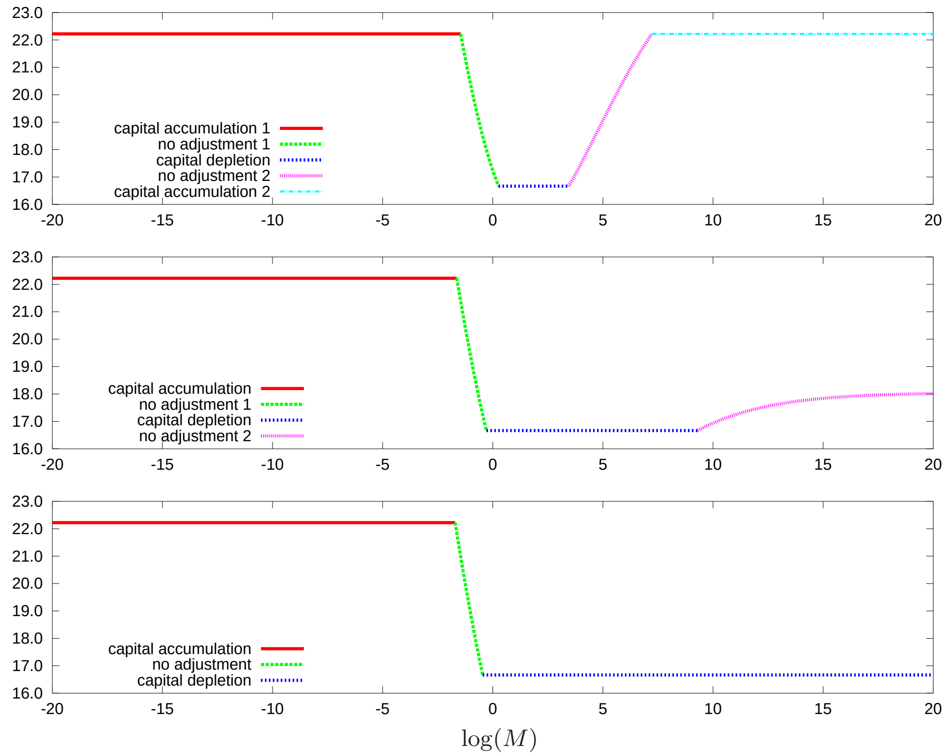

Figure 3: Price-Output Ratio ($\gamma > 1$). The model parameters are set as follows: $\alpha = 0.05$, $\mu - \delta = -2\%$, $\sigma = 3\%$, $\rho_1 = \rho_2 = 5\%$, $\gamma = 3$, $\lambda = 0.5$, $\beta = -1$, $a = 0.1$, and $b = 0.2$. From top to bottom: $\eta = 0.5$, 0.8, 1, respectively.

The value of the aggregate consumption is given in Proposition 3.10. The price-output ratio, $\frac{S_t}{\alpha A_t K_t}$, is $\frac{1}{(1-a)\alpha}$ for the capital accumulation region, $\frac{1}{(1+b)\alpha}$ for the capital depletion region, and depends on M in the no-adjustment region. Figure 3 plots the price-output ratios that correspond to the parameter choices in last three graphs in Figure 1 (for the parameters in the first graph of Figure 1 the price-output ratio is just a constant since the economy always accumulates capital).

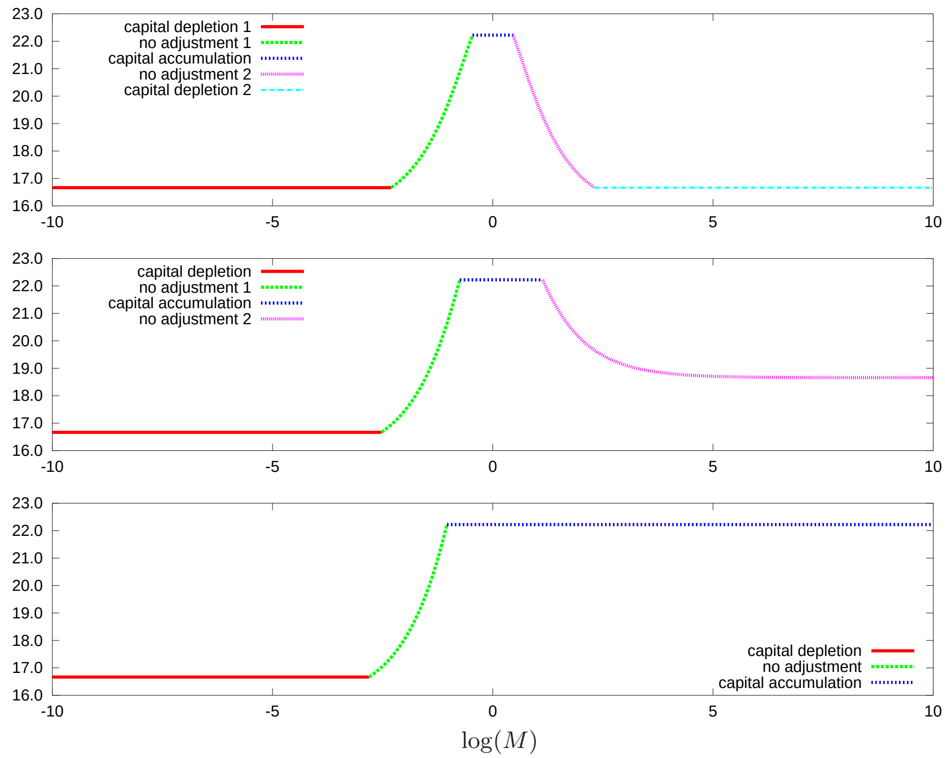

Figure 4: Price-Output Ratio ($\gamma < 1$). The model parameters are set as follows: $\alpha = 0.05$, $\mu - \delta = -2\%$, $\sigma = 3\%$, $\rho_1 = \rho_2 = 5\%$, $\gamma = 0.3$, $\lambda = 0.5$, $\beta = -1$, $a = 0.1$, and $b = 0.2$. From top to bottom: $\eta = 0$, 0.5, 1, respectively.

Figure 4 displays these ratios but now for the parameter choices in the three graphs in Figure 2. In the no-adjustment region speculation tends to decrease the price-output ratio when $\gamma > 1$ and increase the price-output ratio when $\gamma < 1$. This is consistent with Proposition 3.10, where output-price ratio is a weighted average of the output-price ratios from pure exchange economies with homogeneous investors plus an adjustment term. The adjustment term is negative when $\gamma < 1$ and positive when $\gamma > 1$.

As indicated in these figures, the price-output ratio is constant when the economy is in the investment/divestment regions. However, the price-output ratio varies with M in the no-adjustment region. Therefore, the stock price volatility only reflects fundamental risk in the investment/divestment region but can depend on extraneous risk in the no-adjustment region. Speculation does not necessarily cause the stock to be more volatile; at best this can only happen in the no-adjustment region.

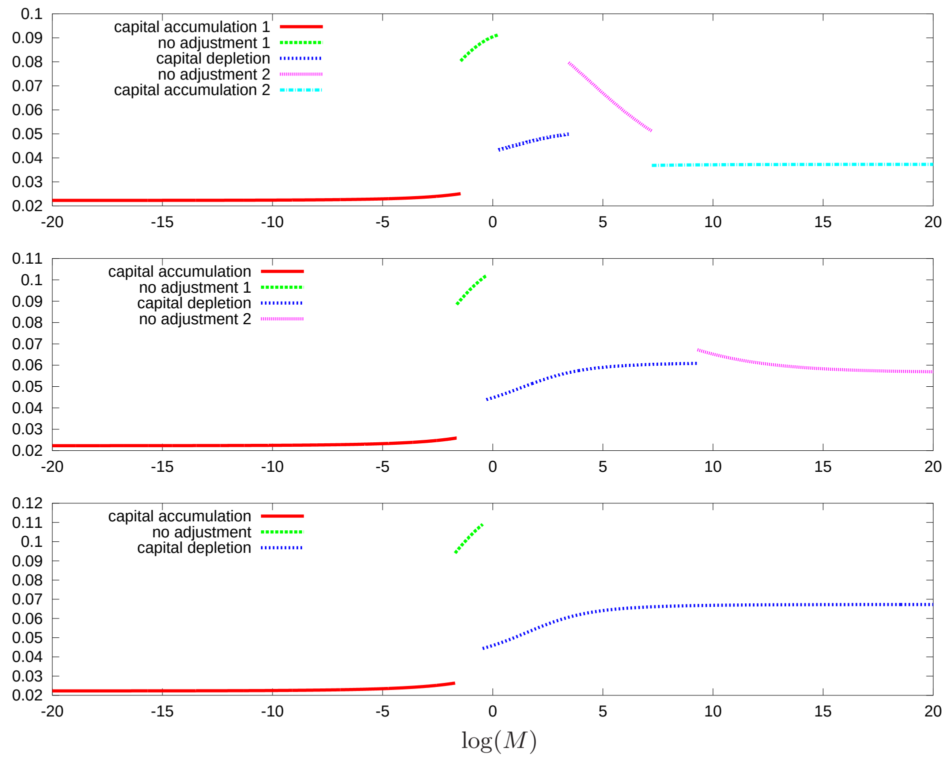

Figure 5: Interest Rate ($\gamma > 1$). The model parameters are set as follows: $\alpha = 0.05$, $\mu - \delta = -2\%$, $\sigma = 3\%$, $\rho_1 = \rho_2 = 5\%$, $\gamma = 3$, $\lambda = 0.5$, $\beta = -1$, $a = 0.1$, and $b = 0.2$. From top to bottom: $\eta = 0.5$, 0.8, 1, respectively.

We can also see clearly the distinct of behavior of the real interest rate across the production and exchange phases. Figure 5 plots interest rate corresponding to the case of $\gamma > 1$ for the parameter choices in the last three panels of Figure 1. First notice that interest rate is not continuous and jumps at the transition points where the economy changes from production to exchange and *vice versa*. As shown in this example, the jump size of interest rate can be very large, which reflects the different nature between a production and an exchange economy. When the economy in the production phase, the interest rate is regulated since production responds to changes in the financial market. However, in the pure exchange phase aggregate consumption does not respond to changes in the financial market and the risk-free rate is solely determined by investors' *local* intertemporal rate of marginal substitution for consumption.

As shown in Proposition 3.9, the high interest rate in the no-adjustment region is due to the β^2 term in Equation (3.49), which is positive for the case of $\gamma > 1$. This term is especially strong when the consumption weights of investors are close to each other and the output doesn't adjust to the financial market. In this case because both investors perceive an attractive future investment opportunity set; all else equal they would tend to increase consumption and the interest rate must go up to clear markets.

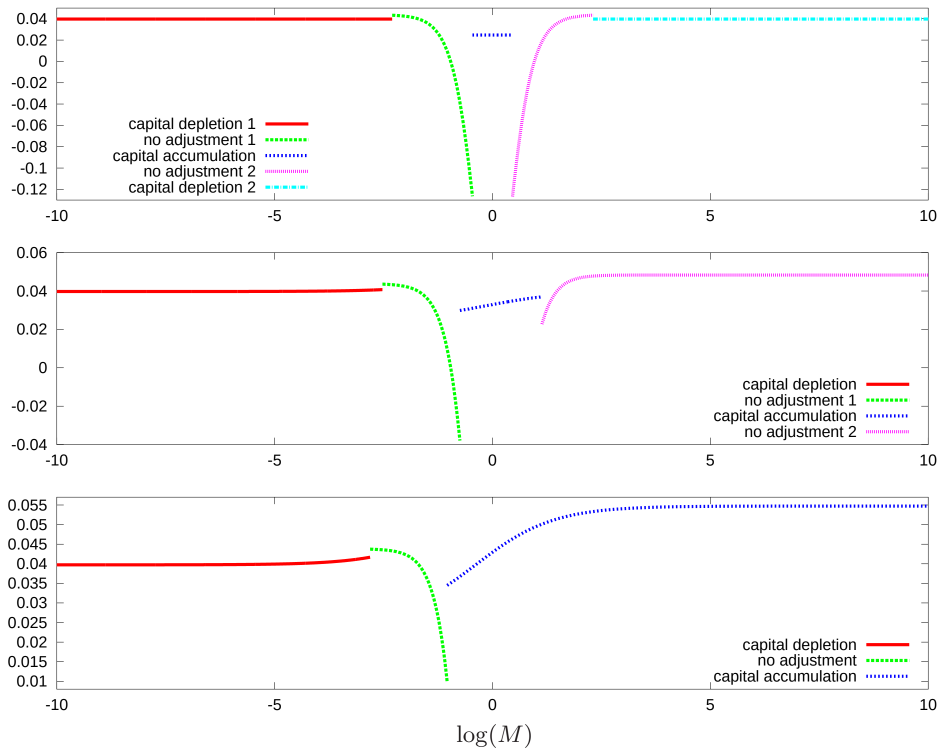

Figure 6: Interest Rate ($\gamma < 1$). The model parameters are set as follows: $\alpha = 0.05$, $\mu - \delta = -2\%$, $\sigma = 3\%$, $\rho_1 = \rho_2 = 5\%$, $\gamma = 0.3$, $\lambda = 0.5$, $\beta = -1$, $a = 0.1$, and $b = 0.2$. From top to bottom: $\eta = 0$, 0.5, 1, respectively.

Figure 6 plots the interest rate when $\gamma < 1$ for the parameter choices in Figure 2. In this case, the β^2 term in Equation (3.49) has a negative impact on the equilibrium interest rate, thus, interest rate can be very low, or even negative, as in the first two panels in Figure 6.

Notice in Figure 6 as η increases from 0 to 1, the real interest rate in an economy with only investor 2 also increases. Changes in η also affect the transition points between the production and exchange phases. The equilibrium interest rate behaves differently in the exchange phase, especially when the consumption weights for investors are close to each other in the exchange region. In this case, both investors have plenty of wealth to exploit the profitable future investment opportunities.

4.3 Extraneous Risk Is Not Tradable

When only the first financial security is traded, extraneous risk is essentially not traded. To illustrate the effects of extraneous risk, we repeat the previous numerical exercises in this case. When $\eta = 0$ and investors disagree only about extraneous risk, then when this risk is not traded, the optimal production is to produce according to the way each investor would run the firm in autarky. Therefore in the first graph in Figure 1 the firm would always accumulate capital and pay out $0.036AK$ to its owners assuming extraneous risk is not traded in the financial market. In the first graph in Figure 2 the same analysis applies only now the firm would deplete capital and pay $0.061AK$ to the owners. Immediately we see the dramatic difference in production that trade in extraneous risk can create.

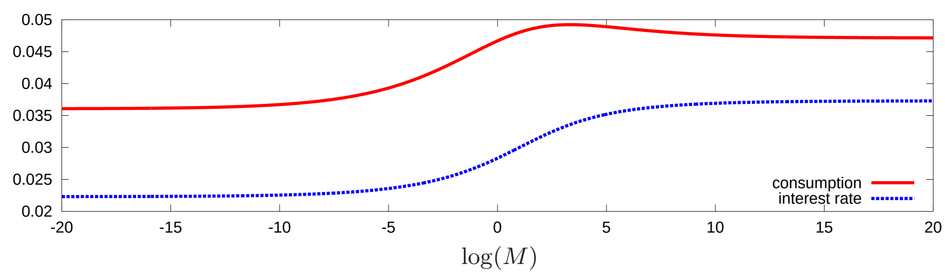

Figure 7: Extraneous Risk Is Not Tradable ($\gamma > 1$). The model parameters are set as follows: $\alpha = 0.05$, $\mu - \delta = -2\%$, $\sigma = 3\%$, $\rho_1 = \rho_2 = 5\%$, $\gamma = 3$, $\lambda = 0.5$, $\beta = -1$, $a = 0.1$, $b = 0.2$, and $\eta = 0.5$.

We now examine cases where there is a mix of disagreement over both fundamental and extraneous risk. The case when $\gamma > 1$ is presented in Figure 7 where we plot the ratio of consumption to AK and the interest rate versus the logarithm of M for $\eta = 0.5$. In this case, the economy is always in the capital accumulation phase and the effects of disagreement are quite muted compared to the case where extraneous risk is traded. When the extraneous risk is traded, the phases of exchange and capital depletion appear as shown in the second panel of Figure 1. The impacts of speculation on extraneous risk may be very substantial as shown by this example; the region that the economy stays in the exchange and depletion phases is very large. Trading extraneous risk also affects the behavior of asset prices profoundly as the price-output ratio is constant and interest rate is smooth and less variable relative to the ones presented in Figures 3 and 5.

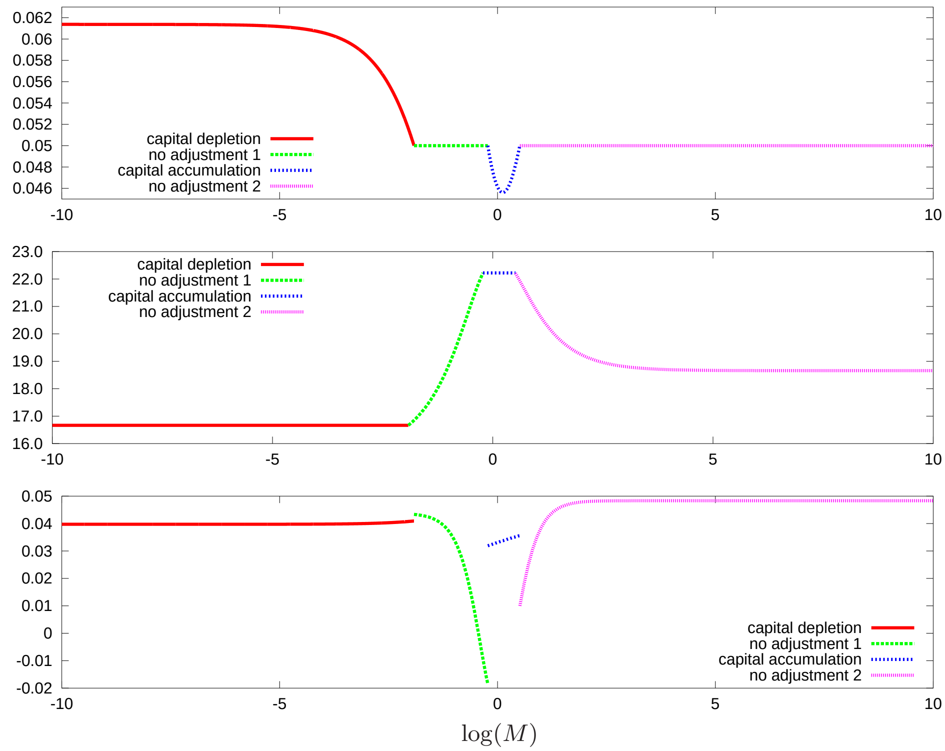

Figure 8: Extraneous Risk Is Not Tradable ($\gamma < 1$). The model parameters are set as follows: $\alpha = 0.05$, $\mu - \delta = -2\%$, $\sigma = 3\%$, $\rho_1 = \rho_2 = 5\%$, $\gamma = 0.3$, $\lambda = 0.5$, $\beta = -1$, $\eta = 0.5$, $a = 0.1$, and $b = 0.2$. From top to bottom: aggregate consumption to AK ratio, price-output ratio, and interest rate, respectively.

Figure 8 plots aggregate consumption to AK, price-output ratio of output, and interest rate versus the logarithm of M for the case of $\gamma < 1$ when extraneous risk is not traded and $\eta = 0.5$. In this example, investor 1 in autarky would deplete capital and investor 2 in autarky would not adjust the capital stock. When there is trade in only fundamental risk, the production also transitions to an exchange phase and capital accumulation phases. In this case, trade in fundamental risk increases investment in the real economy relative to the case when there is no disagreement; this effect is magnified when extraneous risk is also traded as shown by the second panel in Figure 2. The effects of trading extraneous risk on asset prices is also evident when we compare the last two panels in Figure 8 with the second panel in Figures 4 and 6, respectively.

4.4 Heterogeneous Time Preferences

Sometimes short term speculation is blamed for impairing the real economy. One possible reason why time preferences might vary is because intermediaries might invest on the behalf of other investors. In this case their compensation contract might induce intermediaries to focus on shorter horizon results. In this section we examine this possibility by introducing heterogeneous time preferences. Lower time discount parameters will generally raise investment since investors become more patient. However, especially when extraneous risk is traded, it is possible that the optimal production is lower than each individual would choose in autarky when $\gamma > 1$ or higher than each investor would choose in autarky when $\gamma < 1$. This is displayed in Figure 9 which examines the optimal production versus the logarithm of M when investors only disagree about extraneous risk and they can trade the extraneous risk. The top graph assumes that $\gamma = 0.3$ and $\rho_1 = 0.025$ and varies ρ_2.

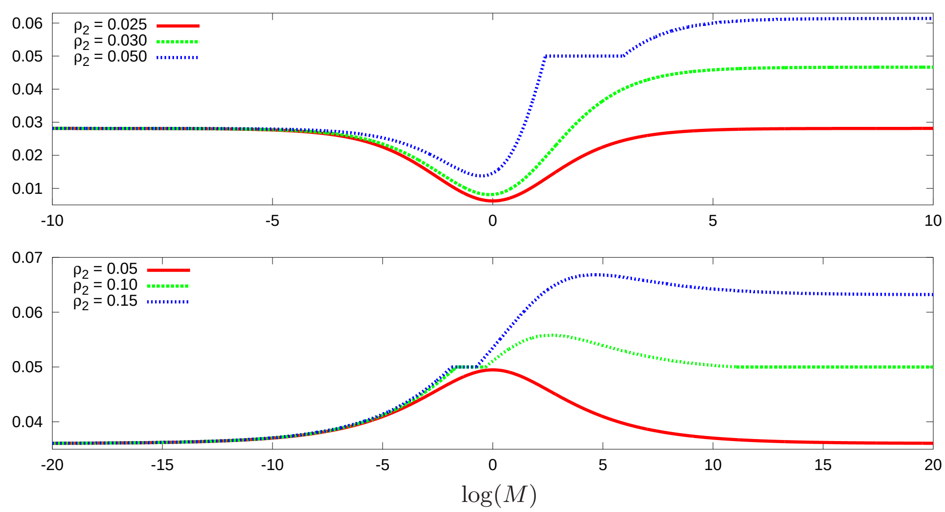

Figure 9: The Effects of Time Preferences. The model parameters are set as follows: $\alpha = 0.05$, $\mu - \delta = -2\%$, $\sigma = 3\%$, $\lambda = 0.5$, $\beta = -1$, $\eta = 0$, $a = 0.1$, and $b = 0.2$; top panel: $\gamma = 0.3$, $\rho_1 = 2.5\%$; bottom panel: $\gamma = 3$, $\rho_1 = 5\%$.

In this case we see that as ρ_2 increases from 0.025 to 0.05, when investor 2 is dominates the economy, then optimal consumption increases and investment thus decreases. However, when both investors are more equally weighted, optimal consumption decreases below that which both investors would choose in autarky. The bottom graph of Figure 9 displays the same information only now for $\gamma = 3$. In this case, the presence of disagreements over extraneous risk tend to work in the same direction as increasing ρ_2 so optimal consumption increases as investor 2 becomes larger and ρ_2 increases.

4.5 Zero Adjustment Costs

In this case, we have $a = b = 0$ and $\bar{c}_{ia} = \bar{c}_{ib}$. Then the no-adjustment regions shrink into points[10] and the ODEs for the adjustment regions in Proposition 3.5 collapse into a single one as

$$\left(\frac{-\Gamma_1}{1-\gamma} + \alpha\right) h(M) + \frac{\beta^2 M^2}{2(1-\gamma)} h''(M) + \left(\frac{\rho_1 - \rho_2}{1-\gamma} - \eta\sigma\beta\right) M h'(M) \\ + \frac{\gamma}{1-\gamma}\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M)^{\frac{1}{\gamma}}\right] h(M)^{1-\frac{1}{\gamma}} = 0,$$

with boundary conditions:

$$\lim_{M\to 0} h(M) = \lambda\left(\frac{\gamma}{\Gamma_1 - (1-\gamma)\alpha}\right)^{\gamma}, \quad \lim_{M\to\infty} \frac{h(M)}{M} = (1-\lambda)\left(\frac{\gamma}{\Gamma_2 - (1-\gamma)\alpha}\right)^{\gamma}.$$

This is the case in which the economy becomes a pure production one, the exchange phase only occurs at isolated points. The transition between capital accumulation and depletion is smooth because h now satisfies the same ODE above in both phases.

Note that the price-output ratio is always $1/\alpha$ when adjustment costs are not present. Figures 10 and 11 repeat the previous numerical exercises without adjustment costs. These plots show that trading financial assets or speculation still has significant impacts on consumption, hence investments, and interest rate. What is interesting is that the adjustment costs do not affect the peak or trough consumption to AK ratios much, although the regions over which these occur narrows in the presence of adjustment costs.

[10] This is not true for the case when $\gamma < 1$ and $\min\{\Gamma_1, \Gamma_2\} \leq (1-\gamma)\alpha$. In this case, the equilibrium should have a no-adjustment region or involves no region of capital adjustment if $\max\{\Gamma_1, \Gamma_2\} \leq (1-\gamma)\alpha$ by Proposition 3.6.

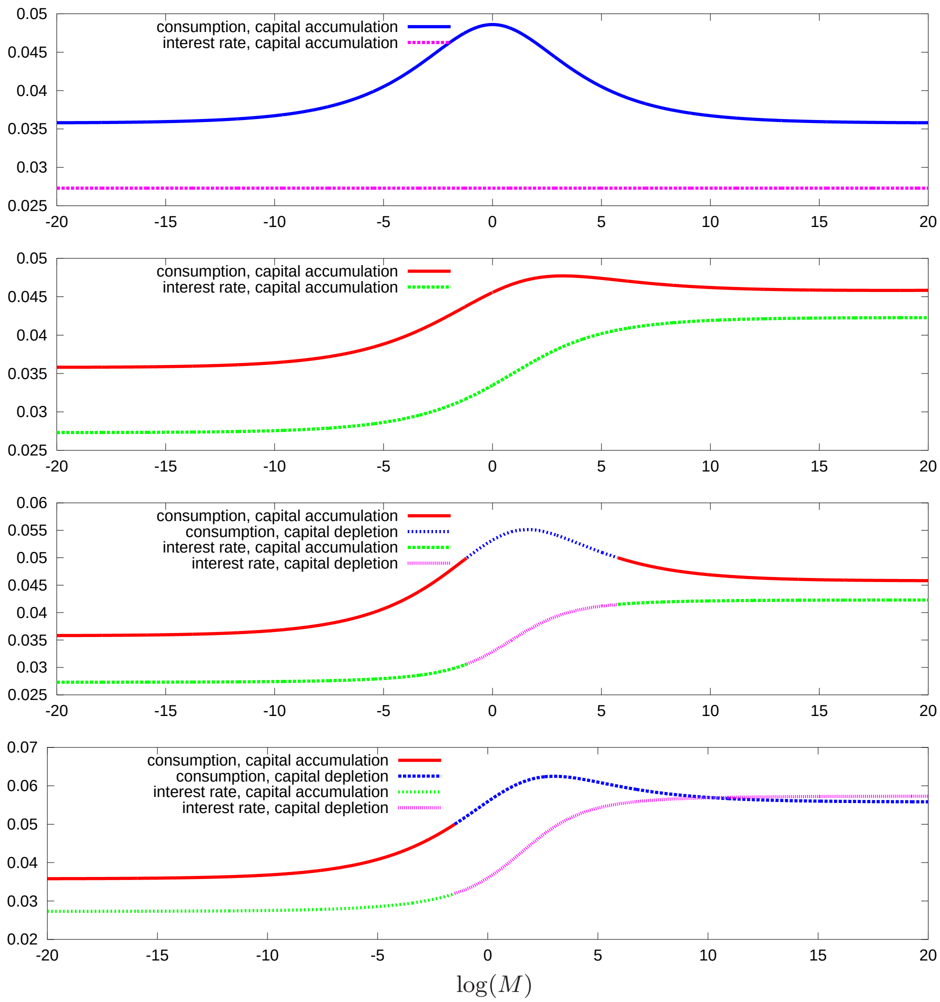

Figure 10: No Adjustment Costs ($\gamma > 1$). The model parameters are set as follows: $\alpha = 0.05$, $\mu - \delta = -2\%$, $\sigma = 3\%$, $\rho_1 = \rho_2 = 5\%$, $\gamma = 3$, $\lambda = 0.5$, $\beta = -1$, $a = 0$, and $b = 0$. From to to bottom: $\eta = 0$, $\eta = 0.5$ (extraneous risk is not traded) , $\eta = 0.5$ (extraneous risk is traded), $\eta = 1$.

The impact of trading extraneous risk is also significant as we compare the second with the third panels in both figures. However, financial markets display less dramatic behavior as the exchange phase never shows up. This indicates that the production always adjusts to changes in financial markets. While trade in financial markets causes dramatic changes in the consumption/investment policy, the financial markets do not display abrupt changes in the dynamics. Contrasting the equilibrium without adjustment costs to the one with adjustment costs highlights the significant role played by this friction.

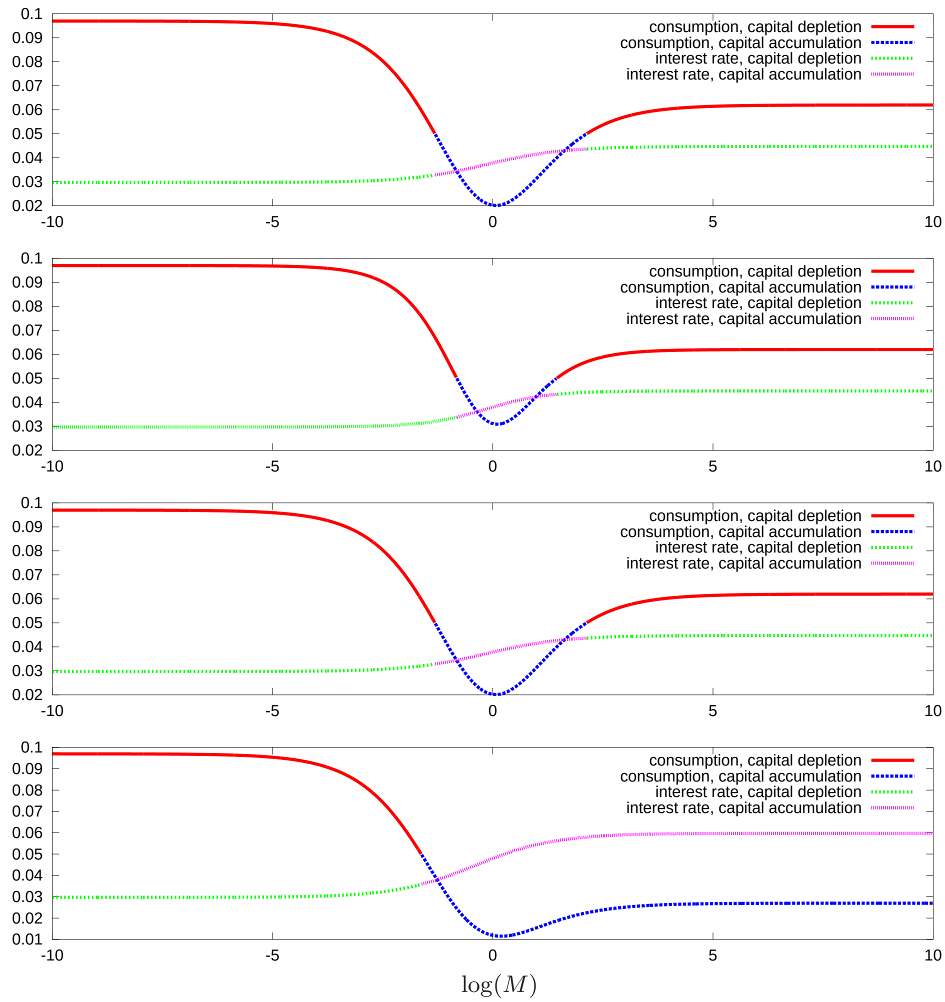

Figure 11: No Adjustment Costs ($\gamma < 1$). The model parameters are set as follows: $\alpha = 0.05$, $\mu - \delta = -2\%$, $\sigma = 3\%$, $\rho_1 = \rho_2 = 5\%$, $\gamma = 0.3$, $\lambda = 0.5$, $\beta = -1$, $a = 0$, and $b = 0$. From top to bottom: $\eta = 0$, $\eta = 0.5$ (extraneous risk is not traded), $\eta = 0.5$ (extraneous risk is traded), $\eta = 1$.

4.6 Infinite Adjustment Costs

When adjustment costs are infinite, both capital accumulation and depletion are not feasible. Thus, there is no adjustment and the exchange phase is the only choice for both investors. This implies that the boundary conditions are

$$\lim_{M \to 0} h(M) = \frac{\lambda \alpha^{1-\gamma}}{\Gamma_1}, \quad \lim_{M \to \infty} \frac{h(M)}{M} = \frac{(1-\lambda)\alpha^{1-\gamma}}{\Gamma_2}.$$

With these boundary conditions, ODE (3.41) admits a closed-form (integral) solution, and hence the value function.

Proposition 4.1. *When adjustment costs are infinite the value function is*

$$V(AK, M) = \frac{(AK)^{1-\gamma}}{1-\gamma} h(M),$$

where $h(M)$ *is given by*

$$h(M) = \left(\frac{2\alpha^{1-\gamma}}{\beta^2(\phi_+ - \phi_-)} \int_M^\infty \frac{\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)x)^{\frac{1}{\gamma}}\right]^\gamma}{x^{\phi_+ + 1}} dx \right) M^{\phi_+} + \left(\frac{2\alpha^{1-\gamma}}{\beta^2(\phi_+ - \phi_-)} \int_0^M \frac{\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)x)^{\frac{1}{\gamma}}\right]^\gamma}{x^{\phi_- + 1}} dx \right) M^{\phi_-}, \quad (4.52)$$

where ϕ_i *for* $i = +, -$ *is given by equation* (6.94) *in the Appendix.*

In this case the economy becomes a pure exchange economy, and value of aggregate output and the real interest rate are the same as that given previously for the region of no adjustment. Obviously, trading financial assets with or without extraneous risk affects the equilibrium by altering the dynamics of M_t.

5 Irreversibility and Housing Market Booms and Busts

Housing construction plays a major role in the economy and is an example of a largely irreversible investment. The recent boom in housing construction followed by an almost complete halt to new construction shows that irreversibility might play a critical role in the economy. In this section we explore the impact of irreversibility. If the adjustment costs for capital depletion are infinite, then investors never deplete capital, thus capital investments become irreversible. One of the implications of irreversibility is that the exchange phase region expands if capital depletion exists in equilibrium when investments are reversible. We reproduce the previous numerical exercises for

$\eta = 0.5$ to illustrate the impacts of irreversible investment for $\gamma < 1$.

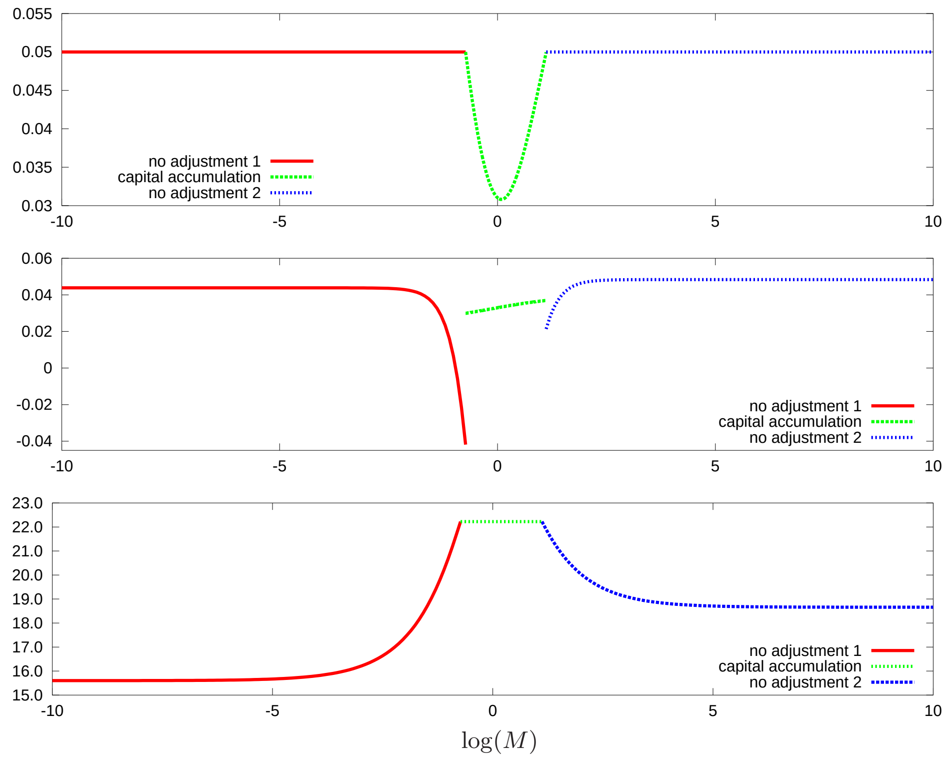

Figure 12: The Effects of Irreversibility ($\gamma < 1$). The model parameters are set as follows: $\alpha = 0.05$, $\mu - \delta = -2\%$, $\sigma = 3\%$, $\rho_1 = \rho_2 = 5\%$, $\gamma = 0.3$, $\lambda = 0.5$, $\beta = -1$, $a = 0.1$, $b = \infty$, and $\eta = 0.5$. From top to bottom: aggregate consumption to AK ratio, interest rate, and asset price-output ratio.

Figure 12 shows how irreversibility affects consumption, the real interest rate, and the price-output ratio. Comparing the first panel of this figure with the second panel of Figure 2, the first capital depletion region is replaced by an exchange region due to irreversibility. Impressively, even though neither investor would choose a production plan which would accumulate capital in autarky, in an equilibrium with both types of investor present, the optimal production would accumulate capital even with irreversibility. Intuitively, in the no-adjustment regions the riskless rate drops and the value of aggregate output rises. The drop in discount rates makes investment attractive so the equilibrium involves capital accumulation. This provides a new insight into the recent housing crisis. Over investment might not depend directly on optimistic investors; instead, speculation in non-fundamental risks can impact discount rates so that investment in housing is becomes optimal, even in the presence of irreversibility. When speculation makes investment in production optimal, the price-output ratio reaches its highest value. When one group of speculators loses enough wealth, investment in production halts, the interest rate jumps down, and the price-output ratio decreases. Thus, speculation can create a boom-and-bust cycle in housing construction and asset prices. Speculation on pure extraneous risk can also generate the boom-and-bust pattern as illustrated in the first panel of Figure 4. In this case, speculation always makes the price-output ratio higher than those in autarky. Interestingly, any sudden change in policy which resolves the source of speculation would lead to collapse in asset prices and production.

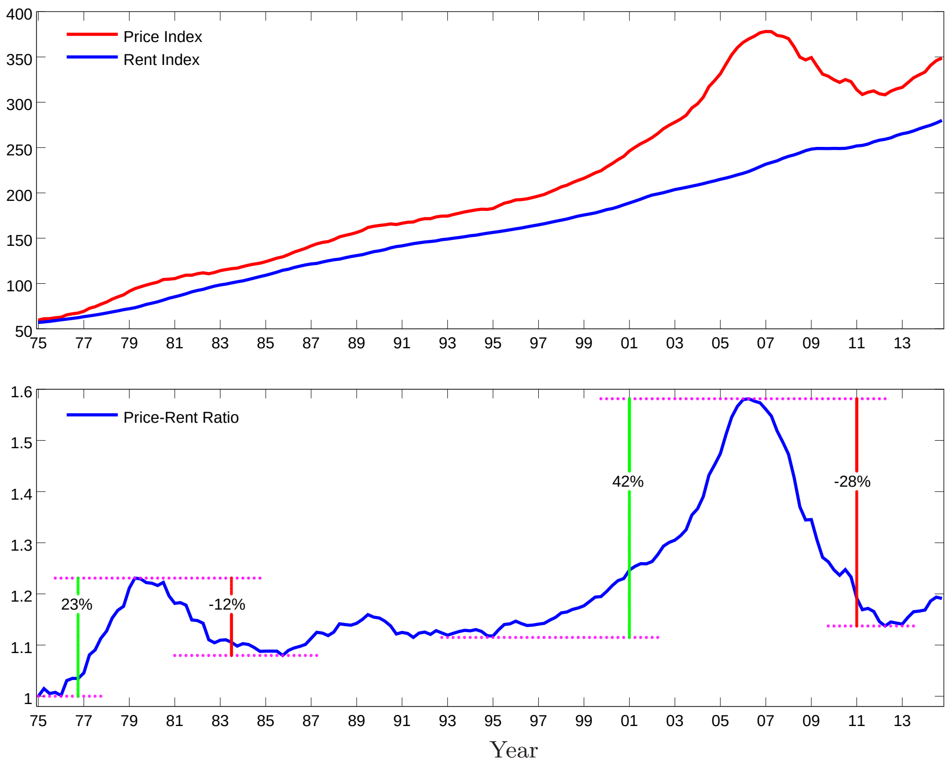

Figure 13: Historical US Housing Price and Rent Indexes: 1975-2014. Both quarterly US housing price index (All-Transactions House Price Index for the United States) and rent index (Consumer Price Index for All Urban Consumers: Rent of primary residence) are downloaded from FRED of St. Louis Fed. The price-rent ratios are normalized by the ratio of the first quarter of 1975. The vertical lines in the second panel indicate percentage increase (green) from trough to peak and decrease (red) from peak to trough.

The first panel of Figure 13 plots the quarterly US housing price index and rent index from 1975 to 2014. The boom and bust in US housing price in the 2000s are very visible. If we examine the price-rent ratio as in the second panel of Figure 13, there are two housing price-rent ratio boom-and-bust cycles during this period, even though the 75-85 cycle does not have a house price boom-and-bust pattern. The boom and bust in price-rent ratio in the 95-12 cycle are about twice of that in the 75-85 cycle. These observed boom-and-bust patterns in housing price-rent ratios can be explained by our model with both heterogeneous beliefs and adjustment costs if we assume that the price-output ratio in the model roughly matches the observed price-rent ratio.[11]

In the example used in Figure 12, by Equation (3.34) we have $\Gamma_1 = 0.064$ and $\Gamma_2 = 0.054$. By Proposition 3.10, the lowest price-output ratio is $1/\Gamma_1 = 15.6$, and the highest price-output ratio is $1/(\alpha(1-a)) = 22.2$. Thus, the highest model trough-to-peak and peak-to-trough percentages are $\frac{22.2-15.6}{15.6} = 42.3\%$ and $\frac{15.6-22.2}{22.2} = -29.7\%$,[12] respectively. These boom and bust sizes are comparable to what we observed in the 95-12 cycle as indicated in Figure 13. The big bust in the price-rent ratio around 2012 suggests that the optimistic investors did not prevail. In the perspective of our model, this suggests that the state variable M goes as follows: starting low, becoming relative high, and ending up low. This dynamics of M can create a boom-and-bust cycle in that the two troughs are relatively close as we observed in the 95-12 cycle. On the other hand, the two troughs in the 75-85 cycle are about 10% different. This asymmetry in troughs can also be understood in our model. As shown in the last panel of Figure 12, the asymmetric troughs are possible if the optimistic investor prevails, that is M goes from low to high, because of $1/\Gamma_2 > 1/\Gamma_1$. This may be one of the key reasons that housing price did not bust even though there was a price-rent boom and bust in the 75-85 cycle.[13]

An alternative explanation to the difference between the two cycles is the role played by extraneous risk that is not correlated with housing market. Under this hypothesis, the 95-12 cycle is mostly affected by the trade in extraneous risk, thus the two troughs are close (with same time preferences and low η). Interestingly, we did witness a unprecedented growth in financial inno-

[11] We can also use the model price-dividend ratio as the observed price-rent ratio. The model price-dividend ratio has qualitatively similar properties regarding the boom-and-bust pattern, even though it is not flat at the top.

[12] These numbers are much larger if we use the model price-dividend ratio as the price-rent ratio because the highest price-dividend ratio is roughly 0.05/0.03 times the price-output ratio in the example.

[13] Obviously, the numerical example does not match the boom and bust sizes of the 75-85 cycle. To do so, we need to adjust the parameter values, for example, the value of β.

vations during the early 2000s. Meanwhile, in the 75-85 cycle, the real risk played a major role, thus, the two related troughs are different. Overall, our model with irreversibility offers reasonable explanations for the key empirical facts that are observed in the housing market.

6 Conclusion

In this paper we examined how speculative trade based on heterogeneous beliefs can affect the real economy. Speculative trade in financial markets can lower or raise investment in capital depending on whether the coefficient of relative risk aversion is greater than or less than one. Interestingly, when investment is irreversible, trade in extraneous risk can lead to higher investment even when investors would choose not to invest when this risk is not traded. This suggests an alternative explanation for the recent housing crisis: trade in extraneous risk affected valuations so as to make investment in housing profitable. When this trade slowed down due to one group of investors losing wealth, the economy reverted to a more normal state.

While any welfare comparisons in the context of the model are likely to be misleading, our model does show that trade based speculative motives can affect real activity through changes in valuations of cash flows. One could, in principal, use these insights to gain insight into how welfare of various participants in the economy might be affected by these changes in real activity stemming from trade in financial markets.

Appendix: Proofs

Proof of Proposition 2.1 Assuming an optimal solution exists, this exercise is pretty standard but we will provide a proof of optimality for investor 2 for the case where only real risk is traded. First observe that $\frac{N_t}{E^1[N_t|\mathcal{F}_t^B]} = \exp\big(-\frac{1}{2}(1-\eta^2)\beta^2 t - \beta\sqrt{1-\eta^2}\hat{B}_t\big)$ and for any feasible choice for $c_{2,t}$ Equation (2.21), Ito's lemma and the solvency condition imply

$$\frac{N_t}{E^1[N_t|\mathcal{F}_t^B]}\xi_t(W_t^2+\theta_2 S_t)+\int_0^t \frac{N_s}{E^1[N_s|\mathcal{F}_s^B]}\xi_s c_{2,s}ds \tag{6.53}$$

is a non-negative local martingale and by Fatou's lemma

$$E^1\left[\int_0^\infty \frac{N_t}{E^1[N_t|\mathcal{F}_t^B]}\xi_t c_{2,t}dt\right] \le \theta_2 S_0. \tag{6.54}$$

Next let

$$c_{2,t}^* = \left(\chi_2 e^{\rho_2 t}\frac{\xi_t}{E^1[N_t|\mathcal{F}_t^B]}\right)^{-\frac{1}{\gamma}}, \tag{6.55}$$

where $\chi_2 > 0$ is chosen to satisfy the static budget constraint (2.26) with equality. Since we can write

$$W_t^2+\theta_2 S_t = \frac{1}{\xi_t}E^1\left[\int_t^\infty \xi_s c_{2,s}^* ds|\mathcal{F}_t\right] \tag{6.56}$$

and $c_{2,t}^*$ depends only on B_t standard martingale representation results imply the existence of a $\phi_{2,t}$ which finances the consumption plan. Then using the inequality $u(c^*)-u(c)\ge u'(c^*)(c^*-c)$ we have

$$\begin{aligned}
E^2\left[\int_0^\infty e^{-\rho_2 t}\left(\frac{c_{2,t}^{*1-\gamma}}{1-\gamma}-\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right)dt\right] &= E^1\left[\int_0^\infty N_t e^{-\rho_2 t}\left(\frac{c_{2,t}^{*1-\gamma}}{1-\gamma}-\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right)dt\right]\\
\ge E^1\left[\int_0^\infty e^{-\rho_2 t}N_t c_{2,t}^{*-\gamma}(c_{2,t}^*-c_{2,t})dt\right] &= E^1\left[\int_0^\infty \chi_2\frac{N_t}{E^1[N_t|\mathcal{F}_t^B]}\xi_t(c_{2,t}^*-c_{2,t})dt\right]\\
&= \chi_2\theta_2 S_0 - \chi_2 E^1\left[\int_0^\infty \frac{N_t}{E^1[N_t|\mathcal{F}_t^B]}\xi_t c_{2,t}dt\right]\ge 0.
\end{aligned}$$

Therefore, we have shown that $c^*_{2,t}$ is optimal choice for investor 2 when only the real risk is traded. Showing optimality for the remaining cases is similar. $\square$

Proof of Proposition 2.2 This follows because, given state prices, maximizing the value of the firm expands both investors budget constraints displayed in Equation (2.19). $\square$

Proof of Proposition 3.1 Given an equilibrium choice of production c^*_t and the consumption choices $c^*_{i,t}$, and any other feasible choice of production c_t and an allocation $c_{1,t}$ and $c_{2,t}$ with $c_{1,t} + c_{2,t} = c_t$,

$$
\begin{aligned}
&E^1\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda\left(\frac{c^{*1-\gamma}_{1,t}}{1-\gamma} - \frac{c^{1-\gamma}_{1,t}}{1-\gamma}\right) + (1-\lambda)M_t\left(\frac{c^{*1-\gamma}_{2,t}}{1-\gamma} - \frac{c^{1-\gamma}_{2,t}}{1-\gamma}\right)\right)dt\right] \\
&\qquad \geq E^1\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda c^{*-\gamma}_{1,t}(c^*_{1,t} - c_{1,t}) + (1-\lambda)M_t c^{*-\gamma}_{2,t}(c^*_{2,t} - c_{2,t})\right)dt\right] \\
&\qquad = E^1\left[\int_0^\infty \xi_t\left(\lambda\chi_1(c^*_{1,t} - c_{1,t}) + (1-\lambda)\chi_2(c^*_{2,t} - c_{2,t})\right)dt\right] \\
&\qquad = \frac{1}{\frac{1}{\chi_1} + \frac{1}{\chi_2}} E^1\left[\int_0^\infty \xi_t(c^*_t - c_t)dt\right] \geq 0,
\end{aligned}
$$

where the first inequality follows from $u(c^*) - u(c) \geq u'(c^*)(c^* - c)$ which is valid for concave functions u, the following equality follows from using the optimal consumption policies in Proposition 2.1, and the final equality comes from setting $\lambda = \frac{\frac{1}{\chi_1}}{\frac{1}{\chi_1}+\frac{1}{\chi_2}}$ and the fact that the optimal equilibrium production maximizes the value of the firm. Therefore, the equilibrium values maximize the planner's objective function. $\square$

Proof of Proposition 3.2 Because Proposition 2.1 indicates that when the second financial asset is not traded, investors will optimally choose consumption adapted to $\mathcal{F}^B_t$, we can restrict our attention to $c_{1,t} \in \mathcal{F}^B_t$ and $c_{2,t} \in \mathcal{F}^B_t$. Notice that for any feasible choice of production c_t and an allocation $c_{1,t}$ and $c_{2,t}$ with $c_{1,t} + c_{2,t} = c_t$ that satisfy $c_{1,t} \in \mathcal{F}^B_t$ and $c_{2,t} \in \mathcal{F}^B_t$

$$
\begin{aligned}
&E^1\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda\frac{c^{1-\gamma}_{1,t}}{1-\gamma} + (1-\lambda)M_t\frac{c^{1-\gamma}_{2,t}}{1-\gamma}\right)dt\right] \\
&\qquad = E^1\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda\frac{c^{1-\gamma}_{1,t}}{1-\gamma} + (1-\lambda)E^1[M_t|\mathcal{F}^B_t]\frac{c^{1-\gamma}_{2,t}}{1-\gamma}\right)dt\right].
\end{aligned}
$$

Using this the proof now is similar to that in Proposition 3.1. □

Proof of Proposition 3.3 This is a special case of Proposition 3.5. The conditions for the capital accumulation, no adjustment, and capital depletion are determined by the following mutually exclusively feasible conditions

$$0 < \bar{c}_{ia} < \alpha, \quad \bar{c}_{ia} \leq 0; \bar{c}_{ia} \geq \alpha; \bar{c}_{ib} \leq \alpha, \quad \bar{c}_{ib} > \alpha,$$

respectively. □

Proof of Proposition 3.4 The form of the value function follows from the homogeneity of the problem. We first show that when $\gamma > 1$ then $h(M)$ is increasing. Observe that for any feasible choice of $c_{1,t}$, $c_{2,t}$, and any $\epsilon > 0$ we have

$$\begin{aligned} V(AK, M) &\geq E^1\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda\frac{c_{1,t}^{1-\gamma}}{1-\gamma} + (1-\lambda)M\frac{M_t}{M_0}\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right)dt\right] \\ &> E^1\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda\frac{c_{1,t}^{1-\gamma}}{1-\gamma} + (1-\lambda)(M+\epsilon)\frac{M_t}{M_0}\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right)dt\right] \end{aligned} \tag{6.57}$$

and taking the supremum in the last line then gives

$$V(AK, M) > V(AK, M+\epsilon).$$

Therefore

$$\frac{h(M)}{1-\gamma} > \frac{h(M+\epsilon)}{1-\gamma}. \tag{6.58}$$

Since $\gamma > 1$, $h(M)$ is increasing. Similarly, we also have

$$\begin{aligned}
MV(AK, M+\epsilon) &\geq E^1\left[\int_0^\infty e^{-\rho_1 t}\left(M\lambda\frac{c_{1,t}^{1-\gamma}}{1-\gamma} + (1-\lambda)M(M+\epsilon)\frac{M_t}{M_0}\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right)dt\right] \\
&> E^1\left[\int_0^\infty e^{-\rho_1 t}\left((M+\epsilon)\lambda\frac{c_{1,t}^{1-\gamma}}{1-\gamma} + (1-\lambda)M(M+\epsilon)\frac{M_t}{M_0}\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right)dt\right] \\
&= (M+\epsilon)E^1\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda\frac{c_{1,t}^{1-\gamma}}{1-\gamma} + (1-\lambda)M\frac{M_t}{M_0}\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right)dt\right] \qquad (6.59)
\end{aligned}$$

and taking supremum yields

$$MV(AK, M+\epsilon) > (M+\epsilon)V(AK, M).$$

This equation is equivalent to

$$\frac{1}{1-\gamma}\frac{h(M+\epsilon)}{M+\epsilon} > \frac{1}{1-\gamma}\frac{h(M)}{M},$$

and hence $h(M)/M$ is deceasing in M. The monotonicity results for the case of $\gamma < 1$ can be proved in a similar manner.

To show $\frac{h(M)}{1-\gamma}$ is convex in M observe for any feasible choice of $c_{1,t}$ and $c_{2,t}$, any $\psi \in [0,1]$, positive M_1 and M_2, we have

$$\begin{aligned}
&E^1\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda\frac{c_{1,t}^{1-\gamma}}{1-\gamma} + (1-\lambda)(\psi M_1 + (1-\psi)M_2)\frac{M_t}{M_0}\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right)dt\right] \\
&\quad = \psi E^1\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda\frac{c_{1,t}^{1-\gamma}}{1-\gamma} + (1-\lambda)M_1\frac{M_t}{M_0}\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right)dt\right] \\
&\quad + (1-\psi)E^1\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda\frac{c_{1,t}^{1-\gamma}}{1-\gamma} + (1-\lambda)M_2\frac{M_t}{M_0}\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right)dt\right] \\
&\quad \leq \psi V(AK, M_1) + (1-\psi)V(AK, M_2). \qquad (6.60)
\end{aligned}$$

Taking the supremum over feasible $c_{1,t}$ and $c_{2,t}$ in the top line of Equation (6.60) then gives

$$V(AK, \psi M_1 + (1-\psi)M_2) \leq \psi V(AK, M_1) + (1-\psi)V(AK, M_2). \tag{6.61}$$

Thus V is convex in M and this implies $\frac{h(M)}{1-\gamma}$ is convex in M.

Finally, Equation (3.38) is a direct implication of the facts that $h(M)$ is increasing in M and $h(M)/M$ is decreasing in M. □

Proof of Proposition 3.5 We want to make the process

$$e^{-\rho_1 t} V(A_t K_t, M_t) + \int_0^t e^{-\rho_1 s} \left(\lambda \frac{c_{1,s}^{1-\gamma}}{1-\gamma} + (1-\lambda) M_s \frac{c_{2,s}^{1-\gamma}}{1-\gamma} \right) ds \tag{6.62}$$

a martingale for the optimal policy and a supermartingale for an arbitrary policy.

The dynamics of AK (3.31) becomes

$$dA_t K_t = \left[A_t K_t(\alpha - \alpha a + \mu - \delta) - (c_{1,t} + c_{2,t})(1-a) \right] dt + A_t K_t \sigma dB_t, \tag{6.63}$$

in the capital accumulation region,

$$dA_t K_t = A_t K_t(\mu - \delta)dt + A_t K_t \sigma dB_t, \tag{6.64}$$

In the no-adjustment region, and

$$dA_t K_t = \left[A_t K_t(\alpha + \alpha b + \mu - \delta) - (c_{1,t} + c_{2,t})(1+b) \right] dt + A_t K_t \sigma dB_t, \tag{6.65}$$

in the capital depletion region. We use subscript K as the indication of taking derivatives with respect to AK. The supermartingale property implies

$$\begin{aligned}- \rho V + V_K\big[AK(\alpha - \alpha a + \mu - \delta) - (1-a)(c_1 + c_2)\big] + \frac{1}{2}V_{KK}\sigma^2(AK)^2 \\ + \frac{1}{2}V_{MM}\beta^2M^2 - V_{KM}\eta\sigma\beta AKM + \left(\lambda\frac{c_1^{1-\gamma}}{1-\gamma} + (1-\lambda)M\frac{c_2^{1-\gamma}}{1-\gamma}\right) \leq 0 \end{aligned} \quad (6.66)$$

in the capital accumulation region,

$$\begin{aligned}- \rho V + V_K\big[AK(\alpha + \alpha b + \mu - \delta) - (1+b)(c_1 + c_2)\big] + \frac{1}{2}V_{KK}\sigma^2(AK)^2 \\ + \frac{1}{2}V_{MM}\beta^2M^2 - V_{KM}\eta\sigma\beta AKM + \left(\lambda\frac{c_1^{1-\gamma}}{1-\gamma} + (1-\lambda)M\frac{c_2^{1-\gamma}}{1-\gamma}\right) \leq 0 \end{aligned} \quad (6.67)$$

in the capital depletion region, and

$$\begin{aligned}- \rho V + V_K AK(\mu - \delta) + \frac{1}{2}V_{KK}\sigma^2(AK)^2 + \frac{1}{2}V_{MM}\beta^2M^2 - V_{KM}\eta\sigma\beta AKM \\ + \left(\lambda\frac{c_1^{1-\gamma}}{1-\gamma} + (1-\lambda)M\frac{c_2^{1-\gamma}}{1-\gamma}\right) \leq 0 \end{aligned} \quad (6.68)$$

in the no-adjustment region; the martingale property implies Equations (6.66), (6.67), and (6.68) hold with equality for an optimal choice of production.

From homogeneity, the value function is given by $V(AK, M) = \frac{(AK)^{1-\gamma}}{1-\gamma}h(M)$. Therefore maximizing Equation (6.66), the optimal consumption is given by

$$c_1^* = \left(\frac{\lambda}{(1-a)h(M)}\right)^{\frac{1}{\gamma}}(AK), \quad c_2^* = \left(\frac{(1-\lambda)M}{(1-a)h(M)}\right)^{\frac{1}{\gamma}}(AK), \quad (6.69)$$

in the capital accumulation region. Thus, the function h satisfies

$$\begin{aligned}\left(-\frac{\rho_1}{1-\gamma}+\alpha-\alpha a+\mu-\delta-\frac{\gamma\sigma^2}{2}\right)h(M)\\ +\frac{\beta^2M^2}{2(1-\gamma)}h''(M)+\left(\frac{\rho_1-\rho_2}{1-\gamma}-\eta\sigma\beta\right)Mh'(M)\\ +(1-a)\left(\frac{1}{1-\gamma}-1\right)\left[\left(\frac{\lambda}{1-a}\right)^{\frac{1}{\gamma}}+\left(\frac{(1-\lambda)M}{1-a}\right)^{\frac{1}{\gamma}}\right]h(M)^{1-\frac{1}{\gamma}}=0.\end{aligned} \tag{6.70}$$

Substituting Γ_1 as defined in Equation (3.34) yields the ODE for the capital accumulation region.

Similarly, in the capital depletion region, the optimal consumption is

$$c_1^*=\left(\frac{\lambda}{(1+b)h(M)}\right)^{\frac{1}{\gamma}}(AK),\quad c_2^*=\left(\frac{(1-\lambda)M}{(1+b)h(M)}\right)^{\frac{1}{\gamma}}(AK), \tag{6.71}$$

and subsequently, the function h satisfies

$$\begin{aligned}\left(-\frac{\rho_1}{1-\gamma}+\alpha+\alpha b+\mu-\delta-\frac{\gamma\sigma^2}{2}\right)h(M)\\ +\frac{\beta^2M^2}{2(1-\gamma)}h''(M)+\left(\frac{\rho_1-\rho_2}{1-\gamma}-\eta\sigma\beta\right)Mh'(M)\\ +(1+b)\left(\frac{1}{1-\gamma}-1\right)\left[\left(\frac{\lambda}{1+b}\right)^{\frac{1}{\gamma}}+\left(\frac{(1-\lambda)M}{1+b}\right)^{\frac{1}{\gamma}}\right]h(M)^{1-\frac{1}{\gamma}}=0.\end{aligned} \tag{6.72}$$

In the no-adjustment region, $c_1+c_2=\alpha AK$, and the sharing rule gives

$$c_1^*=\alpha AK\frac{\lambda^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}+((1-\lambda)M)^{\frac{1}{\gamma}}},\quad c_2^*=\alpha AK\frac{((1-\lambda)M)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}+((1-\lambda)M)^{\frac{1}{\gamma}}}. \tag{6.73}$$

Thus, the function h satisfies

$$\begin{aligned}\left(-\frac{\rho_1}{1-\gamma}+\mu-\delta-\frac{\gamma\sigma^2}{2}\right)h(M)+\frac{\beta^2M^2}{2(1-\gamma)}h''(M)+\left(\frac{\rho_1-\rho_2}{1-\gamma}-\eta\sigma\beta\right)Mh'(M)\\ +\frac{\lambda}{1-\gamma}\left(\frac{\alpha\lambda^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}+((1-\lambda)M_t)^{\frac{1}{\gamma}}}\right)^{1-\gamma}+\frac{(1-\lambda)M}{1-\gamma}\left(\frac{\alpha((1-\lambda)M)^{\frac{1}{\gamma}}}{\lambda^{\frac{1}{\gamma}}+((1-\lambda)M)^{\frac{1}{\gamma}}}\right)^{1-\gamma}=0.\end{aligned} \tag{6.74}$$

In the no-adjustment region, the left hand side of Equation (6.66) should be smaller than that of Equation (6.68). Thus, given any $0 \le \epsilon < 1$, we should have for any $c_i = (1-\epsilon)c_i^*$ for $i = 1, 2$

$$V_K[AK(\alpha - \alpha a) - (c_1 + c_2)(1-a)] + \lambda \frac{(c_1)^{1-\gamma}}{1-\gamma} + (1-\lambda)M\frac{(c_2)^{1-\gamma}}{1-\gamma} \\ \le \lambda \frac{(c_1^*)^{1-\gamma}}{1-\gamma} + (1-\lambda)M\frac{(c_2^*)^{1-\gamma}}{1-\gamma}, \quad (6.75)$$

which can be simplified to

$$h(M)\alpha(1-a)\epsilon \le \frac{\alpha^{1-\gamma}}{1-\gamma}\left(1-(1-\epsilon)^{1-\gamma}\right)\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M)^{\frac{1}{\gamma}}\right]^{\gamma}, \quad (6.76)$$

where we use Equation (6.73). Thus, taking $\lim_{\epsilon\to 0} \frac{1-(1-\epsilon)^{1-\gamma}}{\epsilon} = 1-\gamma$ by the L'Hospital's rule yields

$$h(M)(1-a) \le \alpha^{-\gamma}\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M)^{\frac{1}{\gamma}}\right]^{\gamma}. \quad (6.77)$$

Similarly, given any $\epsilon \ge 0$, we have for any $c_i = (1+\epsilon)c_i^*$ for $i = 1, 2$

$$V_K[AK(\alpha + \alpha b) - (c_1 + c_2)(1+b)] + \lambda \frac{(c_1)^{1-\gamma}}{1-\gamma} + (1-\lambda)M\frac{(c_2)^{1-\gamma}}{1-\gamma} \\ \le \lambda \frac{(c_1^*)^{1-\gamma}}{1-\gamma} + (1-\lambda)M\frac{(c_2^*)^{1-\gamma}}{1-\gamma}, \quad (6.78)$$

which can be simplified to

$$-h(M)(\alpha + \alpha b)\epsilon \le \frac{\alpha^{1-\gamma}}{1-\gamma}\left(1-(1+\epsilon)^{1-\gamma}\right)\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M)^{\frac{1}{\gamma}}\right]^{\gamma}, \quad (6.79)$$

so using the L'Hospital's rule to $\lim_{\epsilon\to 0} \frac{1-(1+\epsilon)^{1-\gamma}}{\epsilon} = -(1-\gamma)$ yields

$$h(M)(1+b) \ge \alpha^{-\gamma}\left[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M)^{\frac{1}{\gamma}}\right]^{\gamma}. \quad (6.80)$$

We now show $V(AK, M) = \frac{(AK)^{1-\gamma}}{1-\gamma}h(M)$ is the value function. To see this, define $V_t =$

$e^{-\rho_1 t}\frac{(A_tK_t)^{1-\gamma}}{1-\gamma}h(M_t)$ and for the optimal choices of $c_{1,t}$ and $c_{2,t}$ we have

$$dV_t = -e^{-\rho_1 t}\left(\lambda\frac{(c^*_{1,t})^{1-\gamma}}{1-\gamma} + (1-\lambda)M_t\frac{(c^*_{2,t})^{1-\gamma}}{1-\gamma}\right)dt + V_t\left((1-\gamma)\sigma dB_t + \beta\frac{h'(M)M}{h(M)}d\hat{B}_t\right)$$
$$= -V_t\frac{c^*_{1,t}+c^*_{2,t}}{A_tK_t}dt + V_t\left((1-\gamma)\sigma dB_t + \beta\frac{h'(M)M}{h(M)}d\hat{B}_t\right). \quad (6.81)$$

From Proposition 3.4 and the form of the optimal consumption, $\frac{c^*_{1,s}+c^*_{2,s}}{A_sK_s}$ is bounded away from 0 and $0 \leq \frac{h'(M)M}{h(M)} \leq 1$. We can write

$$\frac{V_t}{V_0} = \exp\left(-\int_0^t\frac{c^*_{1,s}+c^*_{2,s}}{A_sK_s}ds\right)\times\exp\left(\int_0^t\left\{-\frac{(1-\gamma)^2\sigma^2}{2} + \eta\beta\sigma\frac{h'(M_s)M_s}{h(M_s)} - \frac{1}{2}\left(\frac{h'(M_s)M_s}{h(M_s)}\right)^2\right\}ds\right.$$
$$\left. + \int_0^t(1-\gamma)\sigma dB_s + \int_0^\tau\frac{h'(M_s)M_s}{h(M_s)}\beta d\hat{B}_s\right) \leq e^{-\underline{c}t}N_t, \quad (6.82)$$

where $\underline{c}$ is the lower bound on $\frac{c^*_{1,s}+c^*_{2,s}}{A_sK_s}$ and from the Novikov condition (Corollary 5.5.13 in Karatzas and Shreve (1991)), N_t is a martingale. Therefore $\lim_{t\to\infty}E[V_t] = 0$ and

$$V(AK, M) = E\left[\int_0^t e^{-\rho_1 s}\left(\lambda\frac{(c^*_{1,s})^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{(c^*_{2,s})^{1-\gamma}}{1-\gamma}\right)ds\right] + E[V_t]$$
$$= E\left[\int_0^\infty e^{-\rho_1 s}\left(\lambda\frac{(c^*_{1,s})^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{(c^*_{2,s})^{1-\gamma}}{1-\gamma}\right)ds\right]. \quad (6.83)$$

When $\gamma < 1$, define

$$\tau_n = \inf\left\{t \middle| \left|e^{-\rho_1 t}\frac{(A_tK_t)^{1-\gamma}}{1-\gamma}h(M_t) + \int_0^t e^{-\rho_1 s}\left(\lambda\frac{c_{1,s}^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{c_{2,s}^{1-\gamma}}{1-\gamma}\right)ds\right| \geq n\right\}. \quad (6.84)$$

Then for an arbitrary feasible strategy the process

$$e^{-\rho_1 t\wedge\tau_n}\frac{(A_{t\wedge\tau_n}K_{t\wedge\tau_n})^{1-\gamma}}{1-\gamma}h(M_{t\wedge\tau_n}) + \int_0^{t\wedge\tau_n} e^{-\rho_1 s}\left(\lambda\frac{c_{1,s}^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{c_{2,s}^{1-\gamma}}{1-\gamma}\right)ds \quad (6.85)$$

is a bounded nonnegative supermartingale. Therefore

$$
\begin{aligned}
V(AK, M) &= E\left[\int_0^\infty e^{-\rho_1 s}\left(\lambda\frac{(c_{1,s}^*)^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{(c_{2,s}^*)^{1-\gamma}}{1-\gamma}\right)ds\right] \\
&\geq E\left[\int_0^{t\wedge\tau_n} e^{-\rho_1 s}\left(\lambda\frac{(c_{1,s})^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{(c_{2,s})^{1-\gamma}}{1-\gamma}\right)ds\right] + E\left[e^{-\rho_1 t\wedge\tau_n}\frac{(A_{t\wedge\tau_n}K_{t\wedge\tau_n})^{1-\gamma}}{1-\gamma}h(M_{t\wedge\tau_n})\right] \\
&\geq E\left[\int_0^{t} e^{-\rho_1 s}\left(\lambda\frac{(c_{1,s})^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{(c_{2,s})^{1-\gamma}}{1-\gamma}\right)ds\right] + E\left[e^{-\rho_1 t}\frac{(A_tK_t)^{1-\gamma}}{1-\gamma}h(M_t)\right] \\
&\geq E\left[\int_0^{\infty} e^{-\rho_1 s}\left(\lambda\frac{(c_{1,s})^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{(c_{2,s})^{1-\gamma}}{1-\gamma}\right)ds\right], \quad (6.86)
\end{aligned}
$$

where the first inequality follows from the supermartingale property, the second inequality follows from Fatou's lemma taking limits as $n \to \infty$, the third inequality follows from the monotone convergence theorem and nonnegativity taking limits as $t \to \infty$. So the strategy $c_{1,t}^*$ and $c_{2,t}^*$ is optimal and $V(AK, M)$ is the value function.

Let us now assume $\gamma > 1$ and fix an initial capital stock K. Consider an arbitrary feasible strategy $c_{1,t}$, $c_{2,t}$ with resulting capital stock K_t. We assume that

$$
E\left[\int_0^\infty e^{-\rho_1 t}\left(\lambda\frac{(c_{1,t})^{1-\gamma}}{1-\gamma} + (1-\lambda)M_t\frac{(c_{2,t})^{1-\gamma}}{1-\gamma}\right)dt\right] > -\infty. \quad (6.87)
$$

Now suppose we start with $K+\epsilon$. Then following the strategy $c_{1,t}$ and $c_{2,t}$ until time τ and behaving according to our candidate optimal strategy thereafter is feasible and has capital stock $K_\tau + \epsilon e^{-\delta\tau}$ at time τ. Moreover, $\frac{(A_\tau(K_\tau+\epsilon e^{-\delta\tau}))^{1-\gamma}}{1-\gamma}h(M_\tau) \geq \frac{(A_\tau(\epsilon e^{-\delta\tau}))^{1-\gamma}}{1-\gamma}h(M_\tau)$. Tedious algebra then reveals

$$
\begin{aligned}
\frac{(A_\tau \epsilon e^{-\delta\tau})^{1-\gamma}}{1-\gamma}h(M_\tau) &= \frac{(\epsilon A_0)^{1-\gamma}}{1-\gamma}\exp\left(-\int_0^\tau\left\{\left(1-\frac{h'(M_s)M_s}{h(M_s)}\right)\Gamma_1 + \frac{h'(M_s)M_s}{h(M_s)}\Gamma_2 - \frac{\beta^2M_s^2h''(M_s)}{2h(M_s)}\right\}ds\right) \\
&\times\exp\left(\int_0^\tau\left\{-\frac{(1-\gamma)^2\sigma^2}{2} + (1-\gamma)\eta\beta\sigma\frac{h'(M_s)M_s}{h(M_s)} - \frac{1}{2}\left(\frac{h'(M_s)M_s}{h(M_s)}\right)^2\right\}ds\right. \\
&\left.+\int_0^\tau(1-\gamma)\sigma dB_s - \int_0^\tau\frac{h'(M_s)M_s}{h(M_s)}\beta d\hat{B}_s\right). \quad (6.88)
\end{aligned}
$$

Since $0 \leq \frac{h'(M)M}{h(M)} \leq 1$, the second line is a martingale. Also, when $\gamma > 1$ $h''(M) < 0$. Therefore,

$$\lim_{\tau\to\infty} E\left[\frac{((A_\tau \epsilon e^{-\delta\tau}))^{1-\gamma}}{1-\gamma} h(M_\tau)\right] = 0. \tag{6.89}$$

Therefore for any strategy which is feasible starting with capital K, defining the stopping times τ_n as before we have

$$\begin{aligned}
V(A(K+\epsilon), M) &\geq E\left[\int_0^{t\wedge\tau_n} e^{-\rho_1 s}\left(\lambda\frac{(c_{1,s})^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{(c_{2,s})^{1-\gamma}}{1-\gamma}\right) ds\right] \\
&\quad + E\left[e^{-\rho_1 t\wedge\tau_n}\frac{(A_{t\wedge\tau_n}(K_{t\wedge\tau_n} + \epsilon e^{-\delta t\wedge\tau_n})^{1-\gamma}}{1-\gamma} h(M_{t\wedge\tau_n})\right] \\
&\geq E\left[\int_0^{t\wedge\tau_n} e^{-\rho_1 s}\left(\lambda\frac{(c_{1,s})^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{(c_{2,s})^{1-\gamma}}{1-\gamma}\right) ds\right] + E\left[e^{-\rho_1 t\wedge\tau_n}\frac{(A_{t\wedge\tau_n}\epsilon e^{-\delta t\wedge\tau_n})^{1-\gamma}}{1-\gamma} h(M_{t\wedge\tau_n})\right] \\
&\geq E\left[\int_0^{t} e^{-\rho_1 s}\left(\lambda\frac{(c_{1,s})^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{(c_{2,s})^{1-\gamma}}{1-\gamma}\right) ds\right] + E\left[e^{-\rho_1 t}\frac{(A_t\epsilon e^{-\delta t})^{1-\gamma}}{1-\gamma} h(M_t)\right] \\
&\geq E\left[\int_0^{\infty} e^{-\rho_1 s}\left(\lambda\frac{(c_{1,s})^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{(c_{2,s})^{1-\gamma}}{1-\gamma}\right) ds\right].
\end{aligned} \tag{6.90}$$

Where the first inequality follows from the supermartingale property, the second inequality comes from taking limits as $n \to \infty$, using monotone convergence theorem and dominated convergence theorems, and the third inequality follows from monotone convergence theorem and Equation (6.89). Now letting $\epsilon \downarrow 0$ we see from continuity,

$$\begin{aligned}
V(AK, M) &= E\left[\int_0^{\infty} e^{-\rho_1 s}\left(\lambda\frac{(c^*_{1,s})^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{(c^*_{2,s})^{1-\gamma}}{1-\gamma}\right) ds\right] \\
&\geq E\left[\int_0^{\infty} e^{-\rho_1 s}\left(\lambda\frac{(c_{1,s})^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{(c_{2,s})^{1-\gamma}}{1-\gamma}\right) ds\right].
\end{aligned} \tag{6.91}$$

Therefore the strategy $c^*_{1,t}$ and $c^*_{2,t}$ is optimal and $V(AK, M)$ is the value function. □

Proof of Proposition 3.6 The aggregate consumption (3.44) follows from the proof of Proposition 3.5.

We can show the other expression as follows. For the capital accumulation region, dividing

Equation (3.39) by $\frac{\gamma(1-a)h(M)}{1-\gamma}$ yields

$$-\frac{\Gamma_1-(1-\gamma)\alpha(1-a)}{\gamma(1-a)}+\frac{\beta^2M^2h''(M)}{2\gamma(1-a)h(M)}+\frac{\rho_1-\rho_2-(1-\gamma)\eta\beta\sigma}{\gamma(1-a)}\frac{Mh'(M)}{h(M)}+c_t^*=0,$$

where we use Equation (3.44) to substitute the optimal consumption. Then, the result follows for the region of capital accumulation by recognizing the following identities:

$$\Gamma_1-\Gamma_2=\rho_1-\rho_2-(1-\gamma)\eta\beta\sigma,\quad \bar{c}_{1a}-\bar{c}_{2a}=\frac{\Gamma_1-\Gamma_2}{\gamma(1-a)},$$

where Γ_i and $\bar{c}_{ia}$ are as defined in (3.34) and (3.35). The result for the region of capital depletion follows similarly. □

Proof of Proposition 3.7 This follows immediately when we change the dynamics of the state variable M to

$$dM_t=(\rho_1-\rho_2)M_tdt-\eta\beta M_tdB_t \tag{6.92}$$

and follow the steps in Proposition 3.5. □

Proof of Proposition 3.8 Equation (6.74) can be simplified as

$$\begin{aligned}-\frac{\Gamma_1}{1-\gamma}h(M)+\left(\frac{\rho_1-\rho_2}{1-\gamma}-\eta\beta\sigma\right)Mh'(M)+\frac{\beta^2M^2}{2(1-\gamma)}h''(M)&\\+\frac{\alpha^{1-\gamma}}{1-\gamma}\left[\lambda^{\frac{1}{\gamma}}+((1-\lambda)M)^{\frac{1}{\gamma}}\right]^{\gamma}&=0.\end{aligned} \tag{6.93}$$

The homogeneous solutions to the ODE (6.93) are given by M^{ϕ_i}, $i=+,-$ where ϕ_i are given by

$$\phi_{\pm}=\frac{(1-\gamma)\left(\eta\beta\sigma-\frac{\rho_1-\rho_2}{1-\gamma}\right)+\beta^2/2\pm\sqrt{\left((1-\gamma)\left(\eta\beta\sigma-\frac{\rho_1-\rho_2}{1-\gamma}\right)+\beta^2/2\right)^2+2\beta^2\Gamma_1}}{\beta^2}, \tag{6.94}$$

where Γ_1 is as defined in (3.34). The general solution to the ODE (6.93) is given by Equation (3.46) with two free parameters C_1 and C_2. Therefore, the general solution in the no-adjustment region is given by Equation (3.46) with the conditions (3.42).

At the free boundaries between accumulation and no-adjustment regions, the first inequality in (3.42) holds as equality. Using this equality to substitute $h(M)(1-a)$ in ODE (3.39) yields the exact same ODE (3.41). Thus, the smooth pasting conditions for h and h' at the boundary imply the same h''. Similar argument also works for the free boundaries between depletion and no-adjustment regions. Thus, h is in C^2 for all M. $\square$

Proof of Proposition 3.9 By Equation (2.23) or Equation (2.25) when extraneous risk is not allowed to trade in Proposition 2.1, we have

$$\frac{e^{-\rho_1 t}(c_{1,t}^*)^{-\gamma}}{(c_{1,0}^*)^{-\gamma}} = \xi_t,$$

where $c_{1,t}^*$ is the optimal consumption of investor 1 derived in the proof of Proposition 3.5. The rest of the proposition follows using Equations (6.69), (6.71), and (6.73). $\square$

Proof of Proposition 3.10 Let c_1^* and c_2^* be the optimal consumption processes for each investor. Using the fact that the allocation is Pareto optimal, the value function can be written

$$\begin{aligned} e^{-\rho_1 t}V(A_tK_t, M_t) &= E\left[\int_t^\infty e^{-\rho_1 s}\left(\lambda\frac{(c_{1,s}^*)^{1-\gamma}}{1-\gamma} + (1-\lambda)M_s\frac{(c_{2,s}^*)^{1-\gamma}}{1-\gamma}\right)ds|\mathcal{F}_t\right] \\ &= \frac{\lambda}{1-\gamma}E\left[\int_t^\infty e^{-\rho_1 s}(c_{1,s}^*)^{-\gamma}\left(c_{1,s}^* + c_{2,s}^*\right)ds|\mathcal{F}_t\right] \\ &= e^{-\rho_1 t}\frac{(A_tK_t)^{1-\gamma}}{1-\gamma}h(M_t). \end{aligned}$$

Therefore,

$$S_t = \frac{1}{e^{-\rho_1 t}(c_{1,t}^*)^{-\gamma}}E\left[\int_t^\infty e^{-\rho_1 s}(c_{1,s}^*)^{-\gamma}\left(c_{1,s}^* + c_{2,s}^*\right)ds|\mathcal{F}_t\right] = \frac{(A_tK_t)^{1-\gamma}h(M_t)}{\lambda(c_{1,t}^*)^{-\gamma}},$$

and the result follows from Equations (6.69) and (6.71) for the adjustment regions, and Equation (6.73) for the no-adjustment region. Equation (3.51) follows by substituting the expression of $\frac{S_t}{\alpha A_t K_t}$

for the no-adjustment region into Equation (3.41). □

Proof of Proposition 4.1 It is straightforward to verify $h(M)$ given by Equation (4.52) solves ODE (3.41). Then what remains for the proof is to verify the boundary conditions. By the L'Hospital's rule,

$$\lim_{M\to 0} h(M) = \frac{2\alpha^{1-\gamma}}{\beta^2(\phi_+ - \phi_-)} \lim_{M\to 0} \Bigg(\frac{-\big[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M)^{\frac{1}{\gamma}}\big]^{\gamma} M^{-(\phi_+ + 1)}}{-\phi_+ M^{-(\phi_+ + 1)}} + \frac{\big[\lambda^{\frac{1}{\gamma}} + ((1-\lambda)M)^{\frac{1}{\gamma}}\big]^{\gamma} M^{-(\phi_- + 1)}}{-\phi_- M^{-(\phi_- + 1)}} \Bigg) = \frac{2\alpha^{1-\gamma}}{\beta^2(\phi_+ - \phi_-)} \Big(\frac{\lambda}{\phi^+} - \frac{\lambda}{\phi_-} \Big) = \frac{\lambda\alpha^{1-\gamma}}{\Gamma_1}.$$

Similarly, we have

$$\lim_{M\to\infty} \frac{h(M)}{M} = \frac{2\alpha^{1-\gamma}}{\beta^2(\phi_+ - \phi_-)} \Big(\frac{1-\lambda}{\phi_+ - 1} - \frac{1-\lambda}{\phi_- - 1} \Big) = \frac{(1-\lambda)\alpha^{1-\gamma}}{\Gamma_2}.$$

Thus, both boundary conditions are satisfied. □

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