Financial Network and Systemic Risk – A Dynamic Model

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Abstract

We develop a dynamic model of banking network to study the systemic risk of a financial system arising from its network interconnections. We build on Eisenberg and Noe (2001). The dynamic model developed allows us to study the dynamics of bank defaults. In contrast to the literature, we show that while the possibility of contagion is determined by interconnectedness of the financial network, whether a financial crisis can occur depends on the capital of the banks in the system. We derive an index that forecasts the occurrence of a financial crisis. We then provide an intuitive measure of systemic risk. To illustrate the potential usefulness of our model, we provide an analysis of the system of twenty two German banks. We show how many of the banks are fundamentally weak, where the contagion effect may arise from, how strong the contagion effect is, and how significant the systemic risk is.

Keywords: financial systemic risk, network, dynamic model.

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1 Introduction

It is well understood that an adverse shock to one part of a financial network may give rise to a crisis through the contagion effect (Allen and Gale (2000), Freixas, Parigi, and Rochet (2000) and Eisenberg and Noe (2001)). The financial crisis of 2007-2009 is perhaps the most recent real world example. The peak of the crisis came when Lehman Brothers collapsed on September 15 of 2008. After Lehman Brothers filed for bankruptcy, the U.S. government came in on the next day and provided emergency loan to AIG with the purpose of stopping the contagion and the crisis from spreading further in the financial system. Interestingly, however, prior to the failure of Lehman Brothers, there was already a sequence of events which pointed clearly to the source of the crisis and what might happen if nothing was done to the financial system. First, there was the decline of the real estate market in the first quarter of 2006 which was widely believed to be the trigger of the crisis (Achraya, Philippon, Richardson, and Roubini (2009)); then there was the fall of New Century Financial and Countrywide in mid 2007, which was then followed by the fall of Bear Stearns in March of 2008. The first major intervention by the US government came after the fall of Bear Stearns. In between the collapses of those financial institutions, Citi Group tried to raise new capital on the order 14 billion dollars.[1] Looking back at the sequence of events leading up to the peak of the crisis, one cannot help but wonder what the dynamics of the events tells us about occurrence of the crisis. From the point of view of regulators, it is not enough to know that the goal of intervention is to stop the contagion. It is also important to know when and why to intervene if needed.

While it is difficult to define financial crisis precisely, it is typically described as a widespread instability that impairs the functioning of a financial system (Bisias, Flood, Lo, and Valavanis (2012)). To stress the importance of dynamics, we argue that it is the dynamics of default events that in fact defines crisis. Contagion can indeed spread the failure of one financial institution to another and eventually lead to the failure of a large number of financial institutions. However, if the spread takes a long time and there is no cluster of failures, it is hard to argue that the collective failure of the financial institutions constitutes a financial crisis. From the point of view of maintaining a well functioning financial market, even though such collective failure of financial institutions points to the fundamental weakness of the system, it is not equivalent to the instability of the system that the managers of the system is most concerned about. On the other hand, if a large number of financial institutions all fail in a short span of time, due to contagion or other reasons, it would cause severe difficulty for the normal functioning of the financial market. Then there is clearly a financial crisis that would concern and would likely need the action of the managers of the financial system. The collective failure of a large number of banks in a short span of time is the feature that defines financial crises, the likelihood of which gives rise to systemic risk.

What then are the important factors that drive the dynamics of financial crisis in a banking network? Network connection is one important factor as shown in Allen and Gale (2000), Freixas, Parigi, and Rochet (2000) and Eisenberg and Noe (2001). Interestingly, however, bank capital does not play a separate role in

[1]See Achraya, Philippon, Richardson, and Roubini (2009) for a description of the timing of events.

their static models. This is rather counter intuitive. Citi group tried to raise capital following the collapse of Bear Stearns. Also bank capital figures importantly in Basel accords for bank risk management. We show in this paper that bank capital is another important factor that drives the dynamics of financial crisis in a banking network. In particular, we develop an index based on the capitals of the banks that forecasts the clustering of bank failures.

With the help of the developed index, we then study the systemic risk of a financial network. The systemic risk of a financial network is defined as the probability of widespread failures in a short span of time of financial institutions. As argued in the literature (Hansen (2013)), while it is relatively easy to identify a financial crisis ex post, by the collapse or near collapse of the financial system caused by the sudden occurrence of the defaults of multiple banks, it is important to quantify ex ante the risk of its occurrence. We will provide a quantitative measure of the likelihood of a financial crisis.

The starting point of our model is Eisenberg and Noe (2001), which is now standard in the literature. The financial system is comprised of a set of banks. Banks are interconnected by their liabilities among each other. Each bank's cash inflow comes from its external cash inflow from sources outside the financial system as well as payments from banks inside the system. One bank's failure in making full payment of its liability will affect the cash inflow of other banks, which in turn may affect how much the bank can pay its own liability. In equilibrium, a payment vector is determined which specifies the payment of each of the banks in the financial system. If a component in the vector is less than the total liability of the corresponding bank, the bank is in default. The payment vector provides a characterization of equilibrium defaults in the financial system.

In our model, time is continuous over the interval $[0, \infty)$. In addition to the external cash inflow, each bank is endowed with an initial bank capital. Banks settle their liabilities at each point in time as in the Eisenberg-Noe model. Over time, a bank's capital may rise or fall. The moment when its capital hits below zero, the bank is in default. The default time of each of the banks will be derived. If a bank never defaults, its default time is set at infinity. It should be intuitively clear that in our model, it will be a coincidence that multiple banks will default simultaneously. More commonly, there will be a sequence of times when some of the banks will default. A crisis is a sudden occurrence of multiple defaults that threatens the stability of the financial system. Let η denote a length of a period and q denote the fraction of banks that will default. We will say that the banking system will experience an (η, q) crisis if greater than q fraction of the banks in the system will default in the time interval $[0, \eta]$. Here q should be viewed as a threshold to capture the severity of the defaults. Failure of greater than q fraction of the banks will cause the dysfunction of the banking system. The η is to capture the intensity of the defaults.[2] A small η means a clustering of defaults. Thus in our model a crisis is defined as the eruption of defaults of a significant percentage of banks in the financial

[2] The intensity and the severity can also be affected by market liquidity. When a bank is short of capital to meet its liability, it can sell its assets. If a large number of banks all sell their assets at the same time, or if a bank tries to sell a large amount of an asset, then the proceed received by the banks from the sale will depend on the liquidity of the market. The less liquid the market, the smaller the proceed. Thus market illiquidity in general exacerbates a financial crisis. In this paper we will abstract away the effect of market illiquidity on systemic risk.

system within a short time span. The systemic risk is then defined as the likelihood of the occurrence of an (η, q) crisis.

After setting up the model, we provide a set of conditions that characterizes the time and severity of a crisis. We then study whether the crisis is brought about by the fundamental weakness of the default banks or by contagion (Allen and Gale (2000) and Freixas, Parigi, and Rochet (2000)). Finally, we apply our model to a system of German banks. We first examine whether there are banks in the system that will default without any adverse shocks to the system and their default times. We then subject each bank to individual adverse shock and examine whether the shock to individual banks can lead to contagion in the system. Finally, we run simulations of simultaneous correlated shocks to the banks in the system and examine the systemic risk, i.e., the likelihood of crisis, of the system.

There is a large literature on financial network and its implication for contagion and systemic risk. Rochet and Tirole (1996) point out that lack of monitoring in a system of interconnected banks can be a source of systemic risk. Morris (2000) shows how contagion may arise from local interaction game. Allen and Gale (2000) and Freixas, Parigi, and Rochet (2000) are the first to show how network connection can give rise to financial contagion. Eisenberg and Noe (2001) provide a more formal model for the study of systemic risk arising in a financial network. Since the pioneering study of Allen and Gale (2000), Freixas, Parigi, and Rochet (2000) and Eisenberg and Noe (2001), a sizable literature has since emerged that examines a number of issues related to the implications of network connectedness for systemic risk. Leitner (2005) and Babus (2015) study how a financial network may arise endogenously. Acemoglu, Ozdaglar, and Tahbaz-Salehi (2013) examine the effect of network structure on the stability of a financial network. Cifuentes, Ferrucci, and Shin (2005) argue that market illiquidity of bank assets can exacerbate a crisis.[3] Amini, Cont, and Minca (2011) propose a stress-test framework for evaluating the impact of a macroeconomic shock on the resilience of a banking network to contagion effects. Glasserman and Young (2013) and Amini, Filipovic, and Minca (2013) examine the likelihood and magnitude of contagion. Rogers and Veraart (2013) investigate the effect of the costs of defaults. See also Allen and Babus (2009) for a survey.

In addition to theoretical study on financial systemic risk, there is also a large body of empirical research on contagion effect in the spread of systemic risk. Elsinger, Lehar, and Summer (2006) apply simulation on a data set of the Austrian banking system, and find that correlated exposures of the banks and financial linkages constitute two driving forces behind the systemic risk. Craig and von Peter (2010) show that German banking network has a tiering structure. Sheldon and Maurer (1998), Wells (2002) and Upper and Worms (2004) study financial contagion in the Swiss, England and German banks, respectively. Jorion and Zhang (2010) provide empirical analysis on how credit contagion occurs via direct counter-party effect. Billio, Getmansky, Lo, and Pelizzon (2007) study the financial crisis of 2007, focusing on the interconnectedness

[3]Market illiquidity constitutes the other important channel of contagion, as the market has a limited capacity to absorb asset fire sales. Allen and Gale (2004) and Gorton and Huang (2004) demonstrate how the limited liquidity in the financial markets generates adverse welfare consequences for the whole system in terms of price volatility, bank defaults, and market inefficiency. Brunnermeier and Pedersen (2009), Glâneanu and Pedersen (2007), and Gromb and Vayanos (2003) provide various theoretical models to study the feedback mechanism in fire sale.

among hedge funds, banks, brokers, and insurance companies, and propose five measures of systemic risk based on statistical relations among the market returns of these institutions.

The rest of the paper is organized as follows. Section 2 is a brief review of the Eisenberg-Noe model. The review helps set up the notations and the background for our dynamic model. Section 3 sets up the dynamic model. It highlights the connection and difference between Eisenberg-Noe model and our model. Section 4 provides the solution of our dynamic model. Armed with the solution, Section 5 focuses on the study of financial crises and systemic risk. Section 6 applies the dynamic model developed to the study of twenty two German banks. Finally, Section 7 concludes.

2 The Eisenberg-Noe Model

To start with, consider the setting of Eisenberg and Noe (2001).[4] There is a network of n nodes, indexed by $i \in \mathcal{S} = \{1, \ldots, n\}$. To be concrete, the network models a financial system of n banks. We will refer to each node as a bank. The application of the model is, however, not limited to network of banks. The banks are interconnected via a liability matrix $L = (L_{ij})$, where L_{ij} denotes the liability (or, payment) from bank i to bank j. In Eisenberg and Noe (2001) model (E-N model hereafter), L_{ij} is interpreted as the debt owed by bank i to bank j. In our dynamic model to be described later, it is interpreted as the flow of fund from bank i to bank j. Naturally, it is assumed $L_{ij} \geq 0$ for $i \neq j$, and $L_{ii} = 0$. In addition, bank i has external assets such as mortgage loans and external liabilities such as deposits. Let the vector $\alpha = (\alpha_i)$, with $\alpha_i \geq 0$, denote the external assets of the banks and let the vector $b = (b_i)$ with $b_i \geq 0$ denote the external liabilities of the banks. In this paper, all vectors are row vectors. The sum of its external liability and its liabilities to other banks gives the total liability of a bank:

$$\ell_i = b_i + \sum_{j \neq i} L_{ij}. \tag{1}$$

The total asset of the bank is

$$a_i = \alpha_i + \sum_{j \neq i} L_{ji}. \tag{2}$$

The difference between the asset of bank i and its liability is its equity, $a_i = \ell_i + e_i$. Let $\ell = (\ell_i)$ denote the liability vector. Let

$$p_{ij} = L_{ij}/\ell_i, \qquad i \in \mathcal{S}. \tag{3}$$

be the fraction of bank i's liability to bank j in its total liability, and let $P = (p_{ij})$ denote the *relative* liability matrix. It is assumed that P is a matrix with a spectral radius < 1, in which case $I - P$ is a matrix that is invertible and the inverse is a non-negative matrix (Berman and Plemmons (1979)). When $b_i > 0$ for all i, the assumption is satisfied.

[4] See also Glasserman and Young (2013).

In the network of banks described above, banks make payments to each other and to outside debtors. When a bank, drawing on its assets, is not able to make the full payment, it will default. As the assets of a bank consists of its external assets and the payments it receives from other banks, one bank's default will have impact on the assets side of other banks in the system. Due to the interconnectedness, one bank's default can affect the ability of other banks to make their payments. Thus a central issue for the E-N model is whether all the banks in the network are able to make full payments to each other and to external debtors, and if not, whether there exists a clearing payment vector. Let $\pi = (\pi_i)$ be a vector. It is a clearing payment vector for the system if it is a solution to the following linear system of equations:

$$\pi_i = \ell_i \wedge \big(\alpha_i + \sum_j \pi_j p_{ji}\big), \qquad i \in \mathcal{S}. \tag{4}$$

where $\wedge$ denote the minimum: $x \wedge y = \min\{x, y\}$. The first term on the right hand side is the total liability of bank i. The term inside the brackets is the external asset of bank i plus the total payment that bank i receives from other banks in the system. Thus what equation (4) says is that each bank $i \in \mathcal{S}$ needs to pay its full liability; or, short of that, all of what it will receive from both external and internal sources. In the latter scenario, bank i defaults, and it pays every other bank $j \neq i$ a proportional payment, i.e., according to $p_{ij} = L_{ij}/\ell_i$. Implicit in equation (4) is the assumption of limited liability. In matrix notation, equations in (4) can be rewritten as

$$\pi = \ell \wedge \big(\alpha + \pi P\big). \tag{5}$$

If a clearing vector exists, then the banking network is in equilibrium in the sense that for each $i \in \mathcal{S}$, bank i based on its assets is able to make the payment π_i, the payment is no greater than the amount of full payment ℓ_i, and in the case where a bank is not able to make the full payment, what it pays is exactly equal to the total value of its internal and external assets. In other words, the banking system will settle on that payment vector.

In Eisenberg and Noe (2001), a "fictitious default algorithm" is suggested to solve (5). The algorithm generates a solution that has the following structure. The set $\mathcal{S}$ is partitioned into two subsets: the set of non-default banks, $\mathcal{N} = \{j : \ \pi_j = \ell_j\}$, and the set of default banks, $\mathcal{D} = \{i : \ \pi_i < \ell_i\}$, along with the payment vector π. Thus the solution not only gives us a clearing vector, it also tells us which banks will default and which will not.

The E-N model is a static model. As argued in the introduction, one limitation of the model is its lack of dynamics. One can think of two time points, today, $t = 0$, and the end of period, $t = T$. Conceptually, this future time T could be the next time instant immediately after 0, in which case the solution to (5), including the clearing payment, π, and the partition of all n banks into default banks, $\mathcal{D}$ and non-default banks, $\mathcal{N}$, would be a full characterization of what happens to the banking system. However, should T be substantially away from 0, the solution to (5) would only tell us that at time T, there will be a set of default banks in $\mathcal{D}$ and a set of non-default banks in $\mathcal{N}$ as well as the clearing payment vector π. Eisenberg and Noe (2001) in their paper did show how their algorithm can be used to produce the default sequence. That, however, does not give the true dynamics of the default events. First, it does not provide us with the time intervals between defaults. Given the time span between today and the future time T, we would want to know not only how — in what sequence — the banks in $\mathcal{D}$ go into default, but also at what time points. Do some of the banks default in cluster around some time point? Are the defaults close to today or to the end of the period? In particular, we would want to know that should a bank default at some point, how does its cash flow evolve up to that point and how would the default impact other banks afterwards, and the factors that determine whether the defaults occur in cluster or not. Second, as will be seen later, the default sequence produced by their algorithm actually does not produce the correct default sequence.

3 A Dynamic Model

With those issues mentioned at the end of the last section in mind, we develop in this section a dynamic model of financial network. We try to take the minimum deviation from the E-N model. The only additional components we introduce are the time dimension and bank capital. There is no uncertainty in the model, for example. Adding uncertainty would certainly add more interesting dynamics. A model without uncertainty, however, allows us to show more clearly some of the main factors that determine the default sequence.

3.1 The Setting

To add time dimension to the E-N model, consider the time interval $t \in [0, \infty)$. We maintain the same notations as in E-N model. However, they now have the interpretation as flows. For example, α_i is now the rate of external cash infusion per unit of time. It can arise from the interest payment or prepayment from mortgage loans. And ℓ_i is the rate of the liability bank i promises to pay every unit of time. It can be due to interest payment to depositors and the changes in the borrowings from other banks. For simplicity, we assume that these rates are constant over the time interval.[5] At any point in time $t \in [0, \infty)$, if bank i's total cumulative inflow (of external cash and internal payments received from other banks) plus the initial amount of equity that bank has falls below its cumulative outflow (liability), default occurs. Here, for simplicity, we assume that the equities of all the banks are liquid. They corresponds to liquid bank capital or cash. In the following, we will use bank capital, cash position and equity interchangeably.

Let $z_i(t)$ denote bank i's bank capital at time t. Let $y_i(t)$ denote bank i's cumulative cash shortfall up to time t so that the cumulative payment it has actually made, as opposed to promised, is equal to $\ell_i t - y_i(t)$. Then, the bank capital of bank i evolves over time according to

$$z_i(t) = z_i(0) + \alpha_i t + \sum_j [\ell_j t - y_j(t)] p_{ji} - [\ell_i t - y_i(t)], \qquad i \in \mathcal{S},\ t \in [0, \infty); \tag{6}$$

where $z_i(0) \geq 0$ is the bank i's initial bank capital, the next term on the right hand side is the cumulative external cash inflow over the period $[0, t]$, the third term is the cumulative receivables from other banks in

[5] It is straightforward to introduce time-varying α, ℓ, etc. as long as they are not stochastic. For the issues we focus on in this paper, that addition does not give us much more insight.

the network over the period $[0,t]$, and the last term is the bank's cumulative payment over the period $0,t]$. Note that the expression of the third term on the right side of (6) in particular captures the assumption of equal priority. To complete the specification of the payment and condition for default, we impose the following constraints on $y_i(t)$ and $z_i(t)$:

$$y_i(0)=0, \quad dy_i(t)\geq 0; \quad z_i(t)\geq 0, \quad z_i(t)dy_i(t)=0; \qquad i\in\mathcal{S},\ t\in[0,\infty). \tag{7}$$

The first two constraints above follow directly from the definition of $y_i(t)$: it being a *cumulative* quantity over time; the third one is the limited liability constraint. Since the lowest possible level of bank capital is zero, the last constraint dictates that bank i must make payments to other banks and outside debtors as long as its bank capital is greater than zero ($z_i(t)>0$), that is, $y_i(t)$ must stay flat (the derivative $dy_i(t)=0$) when $z_i(t)>0$.

It will be convenient to rewrite (6) and (7) as follows:

$$z(t)=z(0)+\alpha t-(\ell t-y(t))(I-P), \tag{8}$$

subject to

$$y(0)=0, \quad dy(t)\geq 0; \quad z(t)\geq 0, \quad z(t)dy(t)=0; \qquad t\in[0,\infty). \tag{9}$$

Equation (8) and the constraints (9) together is our dynamic model of financial network. With a slight abuse of notation, we will denote it by $\mathcal{S}=(z(0),\alpha,\ell,P)$. We want to find a solution (z,y) to equation (8) subject to the constraints in (9).

Suppose that the solution to (8) and (9) exists. As $\ell t-y(t)$ is the cumulative actual payment vector, $\ell-y'(t)$ is the vector of the rates of actual payments.[6] Clearly, $\ell_i-y_i'(t)>0$ if and only if bank i is insolvent. Thus the first time when $\ell_i-y_i'(t)>0$, i.e.,

$$\tau_i=\inf\{t:\ t\in[0,\infty),\ \ell_i-y_i'(t)>0\}$$

is the default time of bank i. If $\tau_i<\infty$, then bank i will default at time τ_i. Otherwise, bank i will never default. Thus the vector process, $\ell-y'(t)$, provides the full characterization of the default sequence of banks in the network.

Equation (8) and the constraints (9) together is known as a *Skorohod Problem*, a dynamic linear complementarity problem. Let

$$x(t)=z(0)+\theta t, \tag{10}$$

where

$$\theta=\alpha-\ell(I-P) \tag{11}$$

Note that $\theta_i=\alpha_i-\ell_i+\sum_j \ell_j p_{ji}$, which is the net payment by bank i assuming all banks in the network make their full payment. Thus $x_i(t)$ is the process of bank capital of bank i at time t if all banks in the

[6] The prime of a function, $y'(t)$, is used to indicate the first order derivative of the function y. Noe that the functions y and z are Lipshitz continuous and hence they are absolute continuous and have derivatives almost everywhere. Whenever the derivatives are used for y or z at time t, we assume their derivatives exist at that point.

network make their full payments. It is known (Chen and Yao (2001), §7.2) that, when $I - P$ is invertible, the solution to (8) and (9) uniquely exists. In fact, this unique solution defines a pair of Lipschitz continuous mappings Φ and Ψ such that $z = \Phi(x)$ and $y = \Psi(x)$.

The structure of the solution to the Skorohod Problem, (8) and (9), reveals an important insight about the default sequence of the banks. The solution, driven by the state vector $x(t)$, describes the continuous cash flows through the network of n banks. The routing of the cash flows among the banks is dictated by the "routing" matrix P. In addition, each bank i has an external cash infusion at the rate α_i. Bank i, by contractual obligations, promises a maximal payment at the rate ℓ_i, but it may have to operate at a lower rate if it runs out of its bank capital and there is not enough cash infusion. Thus, the bank capital process of bank i, $z_i(t)$, can be expressed as a sum of two terms:

$$z_i(t) = x_i(t) + y_i(t) - \sum_j y_j(t) p_{ji}.$$

The first term $x_i(t)$ captures the "net cash outflow" assuming all bank make their full payments; the second term $y_i(t) - \sum_j y_j(t) p_{ji}$ then compensates when some of the banks do not make their full payments. Once the process $z_i(t)$ drops down to zero at time τ, say, it will remain at zero ever after. This fact will be confirmed when we solve the Skorohod Problem in the next section. This corresponds to bank i defaulting at time τ, and we want to be able to derive τ. The structure of the solution suggests that the default time depends on bank i's initial capital $z_i(0)$. It also depends on what happens at all other banks, in particular, whether any of them is already in default. In other words, characterizing the default times is more complex, and in return, yields richer information about the system dynamics than simply partitioning the n banks into default and non-default ones, as in the static E-N model. We will elaborate more along this line in the next section.

Before moving on to the analysis of the dynamic financial network set up above, it is worthwhile to comment on the assumption on the matrix $P = (p_{ij})$ and its implications. We assume that $p_{ij} \geq 0$ for all i, j and $\sum_{j=1}^n p_{ij} \leq 1$ and $I - P$ is invertible. A network with such a P is called an *open* network. As p_{ij} is the fraction of the total liability payment per unit of time from bank i to bank j, the quantity, $1 - \sum_{j=1}^n p_{ij}$, is the fraction of the liability payment from bank i to the outside world. A sufficient condition for a network to be open is that $1 - \sum_{j=1}^n p_{ij}$ is strictly positive for all i. Thus if the leakage to the outside world is nonzero for each bank, then the network is open. In our model, if $b_i > 0$ for all i, then obviously $1 - \sum_{j=1}^n p_{ij} > 0$ for all i. In reality, all banks have liabilities other than those to other banks. Thus a banking network is typically an open system.The financial network is *closed* if the matrix P is such that $\sum_{j=1}^n p_{ij} = 1$ for all i. A closed financial network is one in which cash never flows to the outside world. Payments from banks within the network get distributed to banks in the network. As a result, in such a network, while some banks may default, there will always be some banks that will never default. A closely related notion is the surplus set in Eisenberg and Noe (2001). As Eisenberg and Noe (2001) assumes that $\sum_{j=1}^n p_{ij} = 1$ for all i, their surplus sets are closed subnetworks with positive equity values.

3.2 Payments

To gain a better understanding of the dynamic model presented in section 3.1 and its solution, it is helpful to find in our dynamic model the counterpart of the clearing payment vector in E-N model. The clearing payment vector in E-N model is given by the solution π to equation (5), along with the set of default banks, $\mathcal{D}$. As will be seen, the clearing payment vector π_t and the set of default banks, $\mathcal{D}_t$, will both be time varying in our model .

It is useful to start with the receivable flow vector denoted λ_t. In view of (8), we have

$$\begin{aligned} \lambda(t) &= \alpha + (\ell - y'(t))P, && (12)\\ \pi(t) &= \ell - y'(t), && (13)\\ z'(t) &= \lambda(t) - \pi(t). && (14) \end{aligned}$$

Suppose, at some point $t \in [0, \infty)$, some banks are already in default. We split the set of banks $\mathcal{S}$ into the set of default banks, $\mathcal{D}_t$, and the set of non-default banks, $\mathcal{N}_t$. (Here, banks in $\mathcal{N}_t$ should be viewed as not in default *yet*, as of time t.) For $i \in \mathcal{D}_t$, it must be that its bank capital $z_i(t) = 0$ and hence $z_i'(t) = 0$ (since z_i is nonnegative). In addition, as bank i is already in default, $\lambda_i(t)$ must be less than its liability ℓ_i. Because $z'(t) = \lambda(t) - \pi(t)$, we have $\pi_i(t) = \lambda_i(t)$; that is, the clearing payment rate $\pi_i(t)$ must equal the receibale flow rate $\lambda_i(t)$ for the default bank. For the solvent bank, $i \in \mathcal{N}_t$, the bank should not have shortfall, i.e., $y'(t) = 0$, or equivalently, the clearing payment rate $\pi_i(t)$ must equal its liability rate, i.e., $\pi_i(t) = \ell_i$. In short, in equilibrium, if $\mathcal{D}_t$ is the set of default banks at time t, we must have

$$\text{for } i \in \mathcal{D}_t, \qquad z_i'(t) = 0 \text{ or equivalently } \pi_i(t) = \lambda_i(t) \quad \text{and} \quad \pi_i(t) < \ell_i, \qquad (15)$$

$$\text{for } i \in \mathcal{N}_t, \qquad y_i'(t) = 0 \text{ or equivalently } \pi_i(t) = \ell_i.^7 \qquad (16)$$

Partition the matrix P accordingly into four sub-matrices, $P_{\mathcal{D}_t}$, $P_{\mathcal{D}_t,\mathcal{N}_t}$, $P_{\mathcal{N}_t,\mathcal{D}_t}$ and $P_{\mathcal{N}_t}$, respectively, where $P_{\mathcal{D}_t,\mathcal{N}_t}$ consists of elements from P for $i \in \mathcal{D}_t$ and $j \in \mathcal{N}_t$ and so on. Partition λ, π, α, ℓ and $z(t)$ similarly so that $\lambda = (\lambda_{\mathcal{D}_t}, \lambda_{\mathcal{N}_t})$, $\pi = (\pi_{\mathcal{D}_t}, \pi_{\mathcal{N}_t})$, $\alpha = (\alpha_{\mathcal{D}_t}, \alpha_{\mathcal{N}_t})$, $\ell = (\ell_{\mathcal{D}_t}, \ell_{\mathcal{N}_t})$ and $z(t) = (z_{\mathcal{D}_t}(t), z_{\mathcal{N}_t}(t))$, respectively. Then in view of (12)-(16), we obtain in the equilibrium,

$$\begin{aligned} \lambda_{\mathcal{D}_t}(t) &= \alpha_{\mathcal{D}_t} + \lambda_{\mathcal{D}_t}(t)P_{\mathcal{D}_t} + \ell_{\mathcal{N}_t}P_{\mathcal{N}_t,\mathcal{D}_t}, && (17)\\ \lambda_{\mathcal{N}_t}(t) &= \alpha_{\mathcal{N}_t} + \lambda_{\mathcal{D}_t}P_{\mathcal{D}_t,\mathcal{N}_t} + \ell_{\mathcal{N}_t}P_{\mathcal{N}_t}, && (18) \end{aligned}$$

and $\pi_{\mathcal{D}_t}(t) = \lambda_{\mathcal{D}_t}(t)$ and $\pi_{\mathcal{N}_t}(t) = \ell_{\mathcal{N}_t}$. In the above we have argued that if $\mathcal{D}_t$ is the set of default banks, then $\lambda(t)$ and $\pi(t)$ must satisfy relations (15) and (16). On the other hand, (15) and (16) in fact define the set of default banks at time t. Since $P_{\mathcal{D}_t}$, like P, is also a substochastic matrix with spectral radius < 1,

$I_{\mathcal{D}_t} - P_{\mathcal{D}_t}$ is invertible. Solving (17) and (18), we can obtain,

$$
\begin{aligned}
\lambda_{\mathcal{D}_t}(t) &= [\alpha_{\mathcal{D}_t} + \ell_{\mathcal{N}_t} P_{\mathcal{N}_t,\mathcal{D}_t}](I - P_{\mathcal{D}_t})^{-1}, & (19)\\
\lambda_{\mathcal{N}_t}(t) &= \tilde{\alpha}_{\mathcal{N}_t} + \ell_{\mathcal{N}_t} \tilde{P}_{\mathcal{N}_t}, & (20)\\
\pi_{\mathcal{D}_t}(t) &= \lambda_{\mathcal{D}_t}(t), & (21)\\
\pi_{\mathcal{N}_t}(t) &= \ell_{\mathcal{N}_t}, & (22)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\alpha}_{\mathcal{N}_t} &= \alpha_{\mathcal{N}_t} + \alpha_{\mathcal{D}_t}(I - P_{\mathcal{D}_t})^{-1} P_{\mathcal{D}_t,\mathcal{N}_t}, & (23)\\
\tilde{P}_{\mathcal{N}_t} &= P_{\mathcal{N}_t} + P_{\mathcal{N}_t,\mathcal{D}_t}(I - P_{\mathcal{D}_t})^{-1} P_{\mathcal{D}_t,\mathcal{N}_t}. & (24)
\end{aligned}
$$

(In the above and hereafter, we assume the identity matrix I is of the appropriate dimension clearly from its context; for example, the identity matrix I is of dimension $|\mathcal{D}_t|$ in (19).)

In general, $\lambda(t)$ and $\pi(t)$ are time varying, which results from the expansion of $\mathcal{D}_t$ over time. Before providing a dynamic evolution, we note that the static E-N model provides a limiting scenario; that is, $\mathcal{D}_\infty$ would correspond the set of the default banks $\mathcal{D}$ in the E-N model. If $\mathcal{D} = \mathcal{D}_\infty$ is an empty set, then we have a trivial case and no banks will ever default. Otherwise, the network starts with no default banks, i.e., $\mathcal{D}_t$ is an empty set for small t; and the equilibrium clearing payment meets the liability. This continues till one of the banks in $\mathcal{D}$ defaults, or $\mathcal{D}_t$ would include this default bank. Then, the bank capital at the default bank, $z_{\mathcal{D}_t}$, will remain zero, and the receivable flow vector $\lambda(t)$ and the clearing payment vector $\pi(t)$ are computed by (19)-(22). At this point, we would only need to consider the banks remaining in $\mathcal{N}_t$. These banks form a subnetwork with updated external cash flow rate $\tilde{\alpha}_{\mathcal{N}_t}$ and relative liability matrix $\tilde{P}_{\mathcal{N}_t}$. If we set time t as the new time zero, then the new initial bank capital for this subnetwork would be

$$
\tilde{z}_{\mathcal{N}_t}(0) = z_{\mathcal{N}_t}(0) + z_{\mathcal{D}_t}(0)(I - P_{\mathcal{D}_t})^{-1} P_{\mathcal{D}_t,\mathcal{N}_t} + [\tilde{\alpha}_{\mathcal{N}_t} - \ell_{\mathcal{N}_t}(I - \tilde{P}_{\mathcal{N}_t})]t. \tag{25}
$$

It can be shown that this subnetwork would behave exactly the same as the original network. Then, we could apply the above argument to this subnetwork, and iterate until all the banks that will eventually default default, i.e., $\mathcal{D}_t = \mathcal{D}$. The above construction would be a complete descripton of the network dynamics if we could identify $\mathcal{D}_t$ and $\mathcal{N}_t$ for all $t \geq 0$; this will be described by an algorithm in the next section.

The following proposition shows the existence of equilibrium $\lambda(t)$, $\pi(t)$ and $\mathcal{D}_t$, and relates them to the clearing payment vector π and the set of default banks $\mathcal{D}$ in E-N model. Let $\lambda = (\lambda_i)$ denote the solution to the rate equation:

$$
\lambda = \alpha + (\lambda \wedge \ell)P. \tag{26}
$$

By analogy, λ can be viewed is the total received cash flow vector in E-T model. And $\pi = \lambda \wedge \ell$.

Proposition 1 *Suppose that the matrix P has spectral radius less than one. Then*

(i) *The solution* $\lambda(t)$ *to the equilibrium condition (15)-(16) uniquely exists and is given by (19) and (20). The equilibrium clearing payment vector is given by* $\pi(t) = \lambda(t) \wedge \ell$.

(ii) *Let* $\mathcal{D}$ *be the set of default banks and* π *be the clearing payment vector in E-N model. Then* $\lim_{t\to\infty} \mathcal{D}_t = \mathcal{D}$, $\lim_{t\to\infty} \lambda(t) = \lambda$ *and* $\lim_{t\to\infty} \pi(t) = \pi$.

Note that (ii) of Proposition 1 implies that whether a bank i will *eventually* default or not depends only on α, ℓ and P. In other words, the set of banks that will eventually default is the same as that in E-N model. Intuitively, if λ is the solution of (26), then $\lambda_i < \ell_i$ means bank i's total cash inflow falls short of its payments due to others. It will eventually default (assuming T is large enough) regardless of what its initial level of bank capital is and regardless of the status of other banks. Since $\lambda_i < \ell_i$ is equivalent to $\pi_i < \ell_i$, the same can be said about the payment vector π. This observation reveals the key weakness of the E-N model: whether or not a certain bank i will eventually default is perhaps not the complete story, or not even the most relevant one. It is important to know when bank i will default, and how it relates to its (initial) bank capital as well as to the defaults of other banks. These are exactly the kind of information we can obtain from solving the Skorohod Problem in the next section.

4 Sequence of Bank Defaults

In this section, we provide the solution to the Skorohod Problem (8) and (9). The solution will be in the form of an algorithm which can be used to compute the default times of the banks.

4.1 Deriving the Default Times

We first derive the sequence of default times, denoted

$$\tau^{(1)} < \tau^{(2)} < \cdots < \tau^{(d)}, \qquad \text{where} \quad d := |\mathcal{D}|;$$

and $\tau^{(k)}$ denotes the time when the k-th default occurs. Here, in using the strict inequalities, we assume that default occurs one at a time. In a continuous-time setting, this assumption sacrifices virtually no generality, but greatly simplifies exposition and notation alike.

To start with, suppose $z_i(0) > 0$ for all $i \in \mathcal{S}$. Then, $y_i(t) = 0$ for sufficiently small t and for all $i \in \mathcal{S}$; i.e., at the beginning, every bank i is paying its full liability, thanks to the positive initial bank capital. The first bank to default is the bank, say j, that first reaches $z_j(t) = 0$. Also, bank j must be one with a negative net cashflow:

$$\theta_j = \alpha_j + \sum_{i \in \mathcal{S}} \ell_i p_{ij} - \ell_j < 0. \tag{27}$$

Hence, we have

$$\tau^{(1)} = \min_{j \in \mathcal{S}:\, \theta_j < 0} \left\{ \frac{z_j(0)}{-\theta_j} \right\};$$

and $D_{\tau^{(1)}} = \{j\}$ where j is the argument j that achieves min of the above. Bank j is then the first bank to default at time τ_1.

Prior to that point, there is no default and hence $\mathcal{D}_t = \emptyset$ for $0 \le t < \tau^{(1)}$. We have for all $i \in \mathcal{S}$,

$$y_i(t) = 0, \quad z_i(t) = z_i(0) + \theta_i t, \quad t \in [0, \tau^{(1)}).$$

After that point, i.e., for $t \ge \tau^{(1)}$, $z_j(t) = 0$ for $j \in D_{\tau^{(1)}}$ and bank j can only make payment at the rate λ_j. We can then remove j from $\mathcal{S}$ and consider the subnetwork consisting of the remaining banks, i.e., those indexed by $N_{\tau^{(1)}} = \mathcal{S} \setminus D_{\tau^{(1)}}$. In this subnetwork, the external cash flow rate is $\alpha^{(1)}$ given by $\tilde{\alpha}_{\mathcal{N}_t}$ in (23) and the relative liability matrix is $P^{(1)}$ given by $\tilde{P}_{\mathcal{N}_t}$ in (24), with $\mathcal{D}_t$ replaced by D_1 and $\mathcal{N}_t$ replaced by N_1. We treat τ_1 as the new time zero, and update the new initial bank capital as $z^{(1)}(0)$ given by $\tilde{z}^{(1)}_{\mathcal{N}_t}(0)$ in (25) with $\mathcal{D}_t$ replaced by D_1 and $\mathcal{N}_t$ replaced by N_1. Note that $\alpha^{(1)}$ and $z^{(1)}(0)$ are row vectors of dimension $|N_1|$ and $P^{(1)}$ is an $|N_1| \times |N_1|$ matrix.

We then repeat the above procedure to this new subnetwowrk to identify τ_2 and so forth. Specifically, let

$$\theta^{(1)} = \alpha^{(1)} - \ell_{N_1}(I - P^{(1)}).$$

Only the bank $j \in N_1$ with $\theta^{(1)}_j < 0$ may default next. The next default time is given by

$$\tau = \min_{j \in N_1 :\, \theta^{(1)}_j < 0} \left\{ \frac{z^{(1)}_j(0)}{-\theta^{(1)}_j} \right\}.$$

Then $\tau_2 = \tau_1 + \tau$ and $\mathcal{D}_t = D_1$ for $\tau_1 \le t < \tau_2$.

More formally, the algorithm for default times is as follows.

(0) To initialize, set $k = 0$, $c = 0$, $\tau_0 = 0$; $D_0 = \emptyset$, $N_0 = \mathcal{S}$, $\mathcal{D} = D_0$. Set $z^{(0)}(0) := z(0)$ and $\theta^{(0)} := \alpha - \ell(I - P)$.

(1) If $\theta^{(k)} \ge 0$ or $\mathcal{D} = \mathcal{S}$, then set $\mathcal{D}_t = \mathcal{D}$ for all $t \ge \tau_k$, and return $\mathcal{D}$ as the set of all banks that would eventually default, return τ_k as the time as of when all the banks in $\mathcal{D}$ default, and return $d = k$ as the number of the iteration of the algorithm; stop.

Otherwise, compute

$$\tau = \min_{j \in N_k :\, \theta_j < 0} \left\{ \frac{z^{(k)}_j(0)}{-\theta^{(k)}_j} \right\}. \tag{28}$$

Update the newly default set D_{k+1} to be $\arg\min$ of the above, and $N_{k+1} = N_k \setminus D_{k+1}$.

Set $\mathcal{D}_t = \mathcal{D}$ for $c \le t < c + \tau$; set $\tau_{k+1} = c + \tau$. Advance the clock: $c \leftarrow \tau_{k+1}$.

(2) Set

$$\alpha^{(k+1)} = \alpha^{(k)}_{N_{k+1}} + \alpha^{(k)}_{D_{k+1}}(I - P^{(k)}_{D_{k+1}})^{-1}P^{(k)}_{D_{k+1},N_{k+1}}, \tag{29}$$

$$P^{(k+1)} = P^{(k)}_{N_{k+1}} + P^{(k)}_{N_{k+1},D_{k+1}}(I - P^{(k)}_{D_{k+1}})^{-1}P^{(k)}_{D_{k+1},N_{k+1}}, \tag{30}$$

$$z^{(k+1)}(0) = z^{(k)}_{N_{k+1}}(0) + z^{(k)}_{D_{k+1}}(0)(I - P^{(k)}_{D_{k+1}})^{-1}P^{(k)}_{D_{k+1},N_{k+1}} \tag{31}$$

$$+[\alpha^{(k+1)}_{N_{k+1}} - \ell_{N_{k+1}}(I - P^{(k+1)}_{N_{k+1}})]\tau. \tag{32}$$

$$\theta^{(k+1)} = \alpha^{(k+1)} - \ell_{N_{k+1}}(I - P^{(k+1)}_{N_{k+1}}), \tag{33}$$

and set $\mathcal{D} = \mathcal{D} \cup D_{k+1}$.[8]

Update $k \leftarrow k + 1$.

Go to (1).

Note that the above algorithm ends with identifying all the banks that would eventually default $\mathcal{D}$ and then all the banks that would never default, $\mathcal{N} = \mathcal{S} \setminus \mathcal{D}$. This algorithm identifies $\mathcal{D}_t$ and hence $\mathcal{N}_t = \mathcal{S} \setminus \mathcal{D}_t$ for all $t \geq 0$. This allows us to compute $\lambda(t)$ and $\mu(t)$ through (19)-(22).

4.2 Piecewise Linear Solutions

Using the algorithm in Section 4.1, the sequence of default times can be derived. We now use the sequence of default times to construct the solution to (8) and (9). The solution is piecewise linear. The details are provided in the following proposition.

Proposition 2 The solution to (8) and (9) can be characterized as follows. Both z and y are piecewise linear functions, with $d + 1$ pieces, where d is given by the algorithm above. Here, $\mathcal{D} = \{i \in \mathcal{S} : \lambda_i < \ell_i\}$, and λ_i is the solution to the rate equation in (26). These linear pieces are connected by the default epochs, $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_d < \tau_{d+1} := \infty$, together with its corresponding sequential default bank sets $D_0, D_1, \ldots, D_d$, derived from the above algorithm, specifically as follows:

- Let $\mathcal{D}_k$ be the union of $D_0, D_1, \ldots, D_k$ (and hence, $\mathcal{D}_d = \mathcal{D}$). Recall from the above algorithm (Step (2)), $\theta^{(k)}$ is the netflow rate in the k-th iteration. Then for $k = 0, 1, \ldots, d$,

$$y_j(t) = 0, \quad z_j(t) = z_j(\tau_k) + \theta^{(k)}_j(t - \tau_k); \qquad t \in [\tau_k, \tau_{k+1}), \quad j \in \mathcal{S} \setminus \mathcal{D}_k; \tag{34}$$

and

$$y_j(t) = y_j(\tau_k) + (\ell_j - \lambda^{(k)}_j)(t - \tau_k), \quad z_j(t) = 0; \qquad t \in [\tau_k, \tau_{k+1}). \quad j \in \mathcal{D}_k, \tag{35}$$

where $\lambda^{(k)}_j$ is the jth component of $\lambda^{(k)}$ which is given by the equality (19) with $\mathcal{D}_t$ replaced by $\mathcal{D}_k$ and $\mathcal{N}_t$ replaced by $\mathcal{S} \setminus \mathcal{D}_k$.

[8]Note that $\alpha^{(k+1)}$, $z^{(k+1)}(0)$ and $\theta^{(k+1)}$ are the vectors of dimension $|\mathcal{N}^{(k+1)}|$ and $P^{(k+1)}$ is an $|\mathcal{N}^{(k+1)}| \times |\mathcal{N}^{(k+1)}|$ matrix; their coordinates are in the set $\mathcal{N}^{(k+1)}$.

Proposition 2 has several immediate implications. First, the parameter $\theta_i^{(k)}$ is decreasing in k for any i that is not in default yet. This is because as k increases in each iteration, the liability rate of the bank that goes default in the k-th iteration, ℓ_{j_k}, is replaced by the smaller rate λ_{j_k}. Next, for any non-default bank i, $y_i(t) = 0$ for all t; whereas $z_i(t)$ is piecewise linear and increasing, at rate $\theta_i^{(k)}$. But this rate of increase is decreasing over time (i.e., as k increases), as explained above in (i).[9] Hence, overall, $z_i(t)$ is increasing and concave in t. These facts are evident from (34). Lastly, for any bank that defaults (eventually), say, bank j, up to its default epoch, say τ^k, the associated (z_j, y_j) process behaves exactly like that of a non-default bank described in (ii):[10] $y_j(t) = 0$; and $z_j(t)$ is piecewise linear, with a rate that is decreasing over time. This rate may start off being positive, but as it decreases, it will become negative, and at τ_k pulls $z_{j_k}(t)$ down to zero. After that, $z_{j_k}(t)$ will stay at zero, while $y_{j_k}(t)$ increases piecewise linearly. These phenomena are reflected in (34) and (35).

We close this section with a note that due to the uniqueness of the solution to the Skorohod Problem, the proof for the propositions in this section would be a simple but tedious verification that the proposed piecewise linear functions y and z in the proposition give a solution to the Skorohold Problem.

[9] TAN: Where is (i) which you referred here?

[10] TAN: Where is (ii) which you referred here?

5 Financial Crises and Systemic Risk

While the term has been widely used, the precise definition of financial crisis has been hard to find. Occasionally it is used as synonymous to systemic risk. In general, the term is used to refer to a situation of financial instability. In banking, a crisis typically refers to the financial instability that can arise due to the potential failure of a large number of banks that threatens the proper functioning of the banking system. Financial crisis is hard to define because it is difficult to ascertain what kind failure would cause instability. The matter is made worse when panic arises even before the actual failure of banks and causes the dysfunction of the banking system.

Nonetheless, the root of the problem is the failure of banks and that the failure comes with intensity and severity, i.e., in a short period of time and in large number. Thus in our study, we focus on bank failures and define a crisis as intensive and severe bank failures. As argued in the introduction and shown more precisely in this section, financial crisis is intrinsically a dynamic phenomenon. We investigate the condition for the occurrence of financial crisis and the associated systemic risk.

5.1 Financial Crises

To motivate our definition of financial crisis, we begin with three examples. In all three examples, there are ten banks in the system. The ten banks are linked by a chain, but not a cycle. Bank 1 has liability of \$10 per unit of time to bank 2, bank 2 has liability of \$10 per unit of time to bank 3, and so on. In terms of relative liability matrix P, $p_{i,i+1} = 1$, $i = 1, 2, \ldots, 9$, and $p_{ij} = 0$ for all other i and j. None of the ten banks has external cash inflow so that $\alpha_i = 0$ for all $i = 1, 2, \ldots, 10$. None of the first nine banks has external liability either so that $b_i = 0$ and hence $l_i = 1$ for all $i = 1, 2, \ldots, 9$. Bank 10 has an external liability of \$10 per unit of time so that $b_{10} = 10$ and $\ell_{10} = 10$. The only difference across the three examples is the initial bank capital each bank has.

Example 1. Each bank has an initial bank capital of \$5, that is, $z_i(0) = 5$ for all $i = 1, 2, \ldots, 10$. As bank 1 does not have any cash inflow, it can remain solvent only by drawing down its bank capital. In six months, its bank capital will be depleted and it will default. Bank 2 initially receives payment from bank 1 and makes its payment from what it receives. After bank 1 defaults, it will have to rely on its bank capital to remain solvent. It will default in six months after its bank capital is drawn down to zero. As the chain event happens, all banks will default by the end of year 5. The default times are as follows.

$$\tau_1 = 0.5, \quad \tau_2 = 1, \quad \tau_3 = 1.5, \quad \tau_4 = 2, \quad \tau_5 = 2.5, \quad \tau_6 = 3, \quad \tau_7 = 3.5, \quad \tau_8 = 4, \quad \tau_9 = 4.5, \quad \tau_{10} = 5.$$

Example 2. Only bank 1 has an initial bank capital of \$5, that is, $z_1(0) = 5$ and $z_j(0) = 0$ for all $j = 2, \ldots, 10$. The default times are as follows.

$$\tau_1 = \tau_2 = \cdots = \tau_9 = \tau_{10} = 0.5.$$

Example 3. Bank 1 and bank 6 have an initial bank capital of \$5 and \$15, respectively, that is, $z_1(0) = 5$ and $z_6(0) = 15$ and $z_j(0) = 0$ for all $j \neq 1, 6$. Again all banks will default by the end of year 5. The default times are as follows.

$$\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_5 = 0.5, \qquad \tau_6 = \tau_7 = \tau_8 = \tau_9 = \tau_{10} = 2.$$

Figure 1 illustrates the timing of the bank defaults in the three examples. In the figure, the horizontal

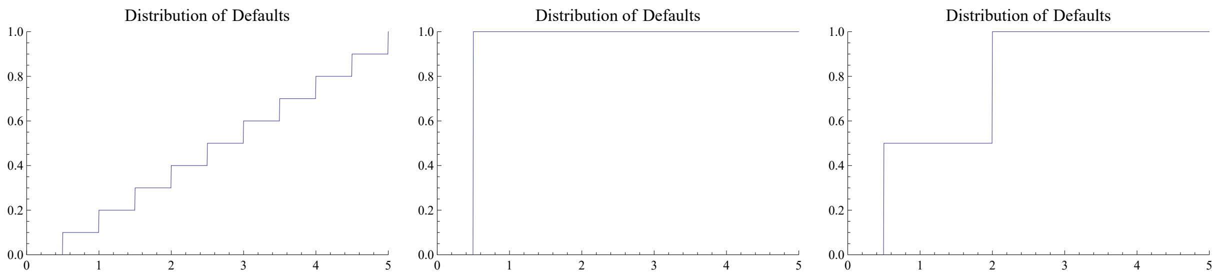

Figure 1: Default Times in Examples 1-3.

axis is time and the vertical axis is the fraction of banks that have defaulted by time t. By year 5, all ten banks default in all three examples. In the first example, the failures of the banks are evenly spread over the five year horizon. In the second example, the timing is highly concentrated around $t = 0.5$. All 10 banks fail around this time. In the third example, 5 banks fail around $t = 0.5$. The other 5 banks fail at $t = 2$. In terms of the severity of bank failures, it is the same across the three example. The failure is 100%. The intensity of bank failure, however, varies across the three examples. By the criteria of intensity and severity, example one is not a case of financial crisis, while the second example is. It seems also reasonable to argue that in the third example there are in fact two financial crises.

The three examples illustrate the importance of both the intensity and the severity of bank failures in the definition of financial crisis. A static model of bank network can easily capture the severity of bank failures, but not their intensity. The three examples also illustrate that bank capital is an important factor that determines the intensity of bank failures.

To provide a definition of financial crisis that incorporate both the intensity and severity of bank failures, it is useful to introduce the default distribution function defined by

$$F(t) = \text{fraction of banks that have defaulted by time } t.$$

Figure 2 is an illustration of a default distribution function. In this figure, there are less than 10% of the banks that have defaulted by the end of year 4 and the defaults occurred evenly over time. Function $F(t)$ on $[0, \infty)$ has the properties of a probability distribution function except that $\lim_{t\to\infty} F(t)$ may be strictly less than one as some of the banks may never default. The difference $F(t') - F(t)$ gives the fraction of banks that went into default in the time interval $[t, t')$. If $F(t)$ is differentiable and $F'(t) = f(t)$, then $f(t)$ can be viewed as default density function. The larger the $f(t)$, the higher the density of default around time t. While the density functions of the three examples do not exist, the big jumps in examples 2 and 3 suggest multiple defaults around certain points in time. More generally, when the density function exists, if the

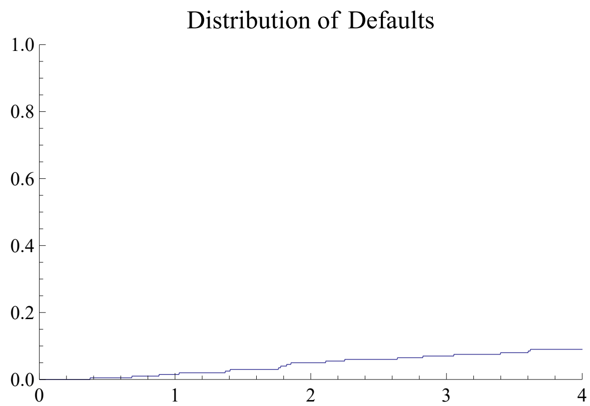

Figure 2: Default Distribution Function.

density function has a single peak with a narrow spread, then defaults are concentrated around the time where the density function is peaked. If the density function has multiple peaks, then there are multiple points in time when defaults concentrate.

Using the default distribution function, we can formally define a financial crisis as follows. Let $\eta > 0$ and $0 < q < 1$.

Definition 3 *A financial crisis with intensity η and severity q, or an (η, q)-crisis for short, is said to have occurred in the time period $[0, T]$ if there exists a time $t \in [0, T - \eta]$ such that $F(t + \eta) - F(t) > q$.*

In this definition, η is the length of a time period and q is a fraction of the banks in the system. $F(t+\eta) - F(t) > q$ means that there is at least q fraction of the banks in the system whose defaults occur in the time interval $[t, t + \eta)$. The greater the q is, the more severe the failures. The shorter the time interval, η, the more intense the failures. Thus a small η and a large q means a major financial crisis in the period $[t, t + \eta]$.

Given a financial system $\mathcal{S}$, how do we know if it is subject to an (η, q)-crisis? Let $\mathcal{S}$ be a financial system and let $\mathcal{D}$ be the set of the banks that will eventually default. For $j \in \mathcal{D}$, let $\tau^{(j)}$ denote the time when bank j defaults. It is to be distinguished from τ_j which is the time of the jth default. The following proposition provides a partial answer to the question.

Proposition 4 *Let $\mathcal{D}'$ be a subset of $\mathcal{D}$. Suppose that the first to default bank i is in $\mathcal{D}'$ and its default time is*
$tau^{(i)} = tau_1 = t_0$. Let θ by given by (11). If $\theta_j < 0$ and $\left|\frac{z_j(0)}{-\theta_j} - \frac{z_i(0)}{-\theta_i}\right| < \eta$ for all $j \in \mathcal{D}'$, then, $\tau^{(j)} \in [t_0, t_0 + \eta)$ for all $j \in \mathcal{D}'$.

Note that $-\theta_j$ is the initial rate of net cash outflow of bank j. At that rate of net outflow, the bank loses θ_j dollars of bank capital per year. Thus, given the initial bank capital $z_j(0)$ and without any disturbance from other banks in the system, bank j can remain solvent for a maximum of $\frac{z_j(0)}{-\theta_j}$ years. After that the bank will not have any bank capital left and hence will default. If there is any disturbance from other banks of the system in the sense that their payments to bank j decrease due to their defaults, the time that bank j can remain solvent will of course be shorter. Thus Proposition 4 says that if the net inflow of a set of banks are all negative, the ratios of initial bank capital over the net outflow are very close to each other and the first to default bank is among them, then the default times of the banks will be very close. If in addition the number of banks in the set is greater than q fraction of the banks in the system, then the financial system will be subject to an (η, q)-crisis around the time t_0.

Proposition 4 states that the closeness of the ratios of initial bank capital over net outflow is an important predictor of whether the financial system is subject to a crisis. It does not provide a full answer to the question raised earlier, as it has not incorporated the contagion effect, which we now turn to.

5.2 Contagion Effect

As argued by Allen and Gale (2000) among others, an important feature of a typical financial crisis is contagion. In this section, we provide a characterization of financial contagion in our model.

We say that bank j as fundamentally weak if its initial net cashflow is negative, that is,

$$\theta_j = \alpha_j + \sum_i^N p_{ij}\ell_i - \ell_j < 0.$$

Clearly when bank j's net inflow $\theta_j < 0$, the bank will bleed to death eventually when its bank capital is run down to zero. This is so even if every bank that owes it money pays in full. It is in this sense we say bank j is fundamentally weak. Define bank, j, as weak by network connection if

$$\theta_j = \alpha_j + \sum_i^N p_{ij}\ell_i - \ell_j \geq 0,$$

but for the payment vector, π,

$$\alpha_j + \sum_i^N p_{ij}\pi_i - \ell_j < 0,$$

where π is the solution of (5). That is, bank j is weak by network connection if its net cashflow is nonnegative when every other bank that owes it money pays in full, but its net cashflow can become negative when some of banks that owe it money do not pay in full. Such a bank operates normally if there is no default in the system, but can be dragged down by the default of other banks in the system. That some banks fail because of the failure of other banks in the network is the contagion effect (Allen and Gale (2000)). We referr to the banks that can be dragged down by the failure of other banks as weak by network connection.

Let $\mathcal{D}_0 \subset \mathcal{D}$ be the set of banks that are fundamentally weak. Then banks in $\mathcal{D}$ but not in $\mathcal{D}_0$ are weak by network connection. An indicator of how severe the contagion effect can be is

$$1 - \frac{|\mathcal{D}_0|}{|\mathcal{D}|} \tag{36}$$

The higher the indicator the great the contagion effect. If $\mathcal{D}_0 = \mathcal{D} = \mathcal{S}$, the indicator is equal to zero, in which case, even though there is severe failure of banks, the failure is caused by the fundamental weakness of the banks in the system and there is no contagion effect. On the other hand, if $|\mathcal{D}_0| = 1$ and $\mathcal{D} = \mathcal{S}$, then the indicator is very high, especially when the number of banks in the system is large. In that case, the banking system is very fragile as the failure of one bank can lead to the failure of all the banks in the system. If the banks all fail at around the same time, then it is a crisis that is contributed mainly by contagion.

It should be emphasized that the notion of contagion in the literature does not have a time dimension to it. As is shown in an earlier example, contagion can be severe and there is no financial crisis. The defaults can spread over time as opposed to being concentrated. As far as it being considered as a measure of contagion effect *in a crisis*, the indicator above is an imperfect measure. To have a measure of the contagion effect in *a crisis*, suppose that we are given an (η, q)-crisis. Let $\mathcal{D}_c$ be the set of default banks in the crisis. Let

$$\text{Contagion Effect Indicator} = 1 - \frac{|\mathcal{D}_0 \bigcap \mathcal{D}_c|}{|\mathcal{D} \bigcap \mathcal{D}_c|} \tag{37}$$

This indicator provides a measure of the contagion effect of the (η, q)-crisis.

Let $\mathcal{S}$ be a financial system and let $\mathcal{D}$ be the set of the banks that will eventually default. The following proposition provides a characterization of crisis with contagion effect.

Proposition 5 *Let $\mathcal{D}'$ be a subset of $\mathcal{D}$. Suppose that the first to default bank i is in $\mathcal{D}'$ and $\tau_i = t_0$. Let θ be given by (11). Let $\mathcal{D}_1 = \{j : \theta_j < 0,\ j \in \mathcal{D}'\}$ and $\mathcal{D}_2 = \mathcal{D}' \setminus \mathcal{D}_1$. Suppose also that for all $j \in \mathcal{D}_2$, $\theta_j = 0$, $z_j(0) = 0$ and $p_{kj} > 0$ for some $k \in \mathcal{D}_1$. If $\left|\frac{z_j(0)}{-\theta_j} - \frac{z_i(0)}{-\theta_i}\right| < \eta$ for all $j \in \mathcal{D}_1$, then $\tau_j \in [t_0, t_0 + \eta]$ for all $j \in \mathcal{D}'$.*

In Proposition 5, $\mathcal{D}_1$ is the subset of banks in $\mathcal{D}'$ that are fundamentally weak and $\mathcal{D}_2$ is the subset of banks that will fail due to contagion effect. Intuitively, for banks that are weak only by network connection, their failure must be due to their connection with weak banks in $\mathcal{D}_1$, that is, $p_{ij} > 0$ for some $i \in \mathcal{D}_1$. However, that is not sufficient. If they have sufficient bank capital or they have large enough positive net cashflows, they may still be able to sustain the impact of the failure of banks in $\mathcal{D}_1$. Short of both, which is what the conditions, $\theta_j = 0$ and $z_j(0) = 0$, try to capture, they will be very vulnerable to the failure of banks in $\mathcal{D}_1$. The failure of banks in $\mathcal{D}_1$ will immediately drag them down as claimed in Proposition 5. Proposition 5 is in the same spirit of Proposition 4, showing that the closeness of the ratios of initial bank capital over net outflow, for the fundamentally weak banks, is an important predictor of whether the financial system is subject to a crisis.

5.3 Amplification Effect

To understand a financial crisis, it is important to understand the channel of magnification effect. It is equally important if not more important to understand the magnitude of the amplification effect. There is a sizable literature on the amplification effect through the leverage/collateral channel (Brunnermierer and Pedersen etc.). In the literature on the network, however, the magnitude of the amplification has not been fully explored. In this section, we provide a quantitative measure of the potential magnitude of the amplification effect.

Now suppose that there is a shock to the cashflow from the external assets, i.e., a shock to α. How will the shock affect the payment vector π? Consider first the final effect. That is, all the banks that will default eventually default have defaulted. In this case, $\pi(\alpha)$ as a function of α is given by $\ell \wedge \lambda(\alpha)$ where $\lambda(\alpha)$ is given by (19) and (20) (with $\mathcal{D}_t$ replaced by $\mathcal{D}$ and $\mathcal{N}_t$ replaced by $\mathcal{N}$). Differentiating yields

$$\frac{\partial \pi_{\mathcal{D}}(\alpha + \delta\xi)}{d\delta} = \xi_{\mathcal{D}}(I - P_{\mathcal{D}})^{-1}$$

This derivative is the magnification effect of the shock to α in the direction of ξ. Then the maximum magnification effect appropriately normalized is

$$M := \max_{\xi \neq 0} \frac{\|\xi(I - P_{\mathcal{D}})^{-1}\|}{\|\xi\|} = \|(I - P_{\mathcal{D}})^{-1}\|$$

which can be viewed as the multiplier of the system.

Now in our dynamic model, bank defaults occur over time. Accordingly, the multiplier changes over time. The best way to understand it is as follows. Initially, as banks start with positive equities, even the fundamentally weak banks will be able to make full payments. It is as if the banks have larger cash flows from external assets, except that the additional cash flows from running down bank equity. Let us assume that ξ_i is the rate at which bank i runs down its equity just so that it can make its payment. That is

$$\ell = \ell \wedge \big(\alpha + \xi + \ell P\big) = \alpha + \xi + \ell P. \tag{38}$$

Thus

$$\ell = (\alpha + \xi)(I - P)^{-1}.$$

When banks run out of equity, ξ will change. It is as if α changes. This change occurs over time. At each point in time, we can examine the amplification effect, which summaries in the multipliers, M_1, M_2, ..., M_d, where $M_d = M$. [11]

5.4 Systemic Risk

In this section we examine the systemic risk that may arise when the fundamentals of banks in the system $\mathcal{S}$ are hit by random shocks.

There are many approaches to modeling random shock to the fundamental of banks in the system $\mathcal{S}$. One general approach, for example, is to model the cash flows as stochastic processes and the liability matrix as a matrix with stochastic components. In this paper, however, we adopt a simpler approach and leave the more general approach to future study. We focus on the random shocks to banks' cash inflows from external assets and their cash outflows due to external liabilities. We can also add shocks to banks assets. As will be shown below, however, the impact of this type of shocks is not as important as one initially thinks.

What we have in mind is the following scenario. Given that shocks constantly hit the system, the regulatory authority or risk manager of the banking system would like to know what the likelihood is that

[11] TAN: I am not sure that I understand this last paragraph.

in the near future there will be a financial crisis which will impair the normal functioning of the banking system. We define the systemic risk of an (η, q)-crisis in the time period $[0, T]$ as the probability of the occurrence of an (η, q)-crisis during the time period. We will examine below the likelihood of the occurrence of an (η, q)-crisis.

Suppose that ϵ is a random vector that represents the shock to the external cash inflow vector α. After the shock, the external cash inflow vector becomes $\alpha + \epsilon$. Similarly, let δ be a random vector that represents the shock to the external liability vector b so that after the shock the external liability vector becomes $b + \delta$. Clearly, after the shocks, all the endogenous variables of the system will be dependent on the shocks. For example, the set of banks that will eventually default, $\mathcal{D}(\epsilon, \delta)$, will depend on the shocks. So will the net cash flow vector, $\theta(\epsilon, \delta)$ and so on.

Given the shocks, let $\mathcal{D}(\epsilon, \delta)$ be the set of the banks that will eventually default. Let $\mathcal{D}_1(\epsilon, \delta)$ be a subset of $\mathcal{D}(\epsilon, \delta)$ such that for all $j \in \mathcal{D}_1(\epsilon, \delta)$, $\theta_j(\epsilon, \delta) < 0$. Suppose further that the first to default bank is in $\mathcal{D}_1(\epsilon, \delta)$ and,

$$\tau(\epsilon, \delta) = \min_{j \in \mathcal{S}: \theta_j(\epsilon, \delta) < 0} \left\{ \frac{z_j(0)}{-\theta_j(\epsilon, \delta)} \right\}.$$

If the shocks (ϵ, δ) are such that

$$\text{prob}\left(\left| \frac{z_j(0)}{-\theta_j(\epsilon, \delta)} - \tau(\epsilon, \delta) \right| < \eta, \text{ for all } j \in \mathcal{D}_1(\epsilon, \delta), \text{ and } |\mathcal{D}_1(\epsilon, \delta)|/|\mathcal{S}| \geq q \right) \geq p$$

then the probability that there is an (η, q)-crisis is greater than p. That is, the systemic risk is greater than p.

The actual computation of the probability is difficult. However, given (α, ℓ, P) and the distribution function of (ϵ, δ), defaults can be simulated. The systemic risk can then be assessed numerically. We will illustrate this in the next section.

6 A Study of Twenty Two German Banks

In this section, we provide an analysis of the systemic risk of a network of twenty two German banks to illustrate the potential application of the model developed in this paper. The analysis is based on the information that can be obtained from the financial statements of the twenty two banks. In the past few decades, there is a declining trend in the number of banks in Germany. Germany had over 3,000 banks in 1980s while it has about 1,800 banks as of 2015, a decline of over 40%. The twenty two banks we base our study on are public banks and are on the larger side in term of size. They include commercial banks, savings banks, cooperatives, which are the three pillars in the German banking system, as well as special banks. Overall, they account for more than 50% of the total assets and liabilities of German banks.

6.1 Preliminary Analysis

Ideally, to study the network of twenty two banks, we would like to have accurate information on the flows of funds inside the network, that is, the flows among the banks themselves, as well as the flows of funds between the banks and parties outside the bank network. However, information at such detailed level is practically impossible to obtain. Thus we need to estimate the flows. The approach followed in this section is to use the information from the balance sheets of banks, coupled with assumptions on the structure of the network, to infer the flows of funds. In practical applications, if additional information can be obtained, the approach can be suitably adjusted and the accuracy of the estimates can be significantly improved.

Let the total claims of bank i on other banks in the network be A_i^b and the total other claims of the bank be A_i^o so that the total asset of the bank is $A_i = A_i^b + A_i^o$. Similarly, let the total liability of bank i to other banks in the network be D_i^b and the total other liabilities of the bank be D_i^o so that the total liability of the bank is $D_i = D_i^b + D_i^o$. The equity of the bank is then $E_i = A_i - D_i$. We assume that E_i is completely liquid so that it can be viewed as the cash position of the banks in the model. When the equity of the bank runs down to zero, the bank is bankrupt. The asset A_i, liability D_i and E_i of the bank are stock values. To estimate the flows of funds, we make two assumptions. The first assumption is that the flow of funds from bank i to bank j in the rest of the network is a fixed percentage k_{ij}^b of the total liability of bank i to the rest of the network. Implicit in this assumption is the assumption that the network is a complete network. That is, there is a net flow of fund between every pair of banks in the network. The second assumption is that the flow of funds received by bank i from outside the network is a fixed percentage k_i^o of the total other claim of bank i. By these two assumptions, $\alpha_i = k_i^o A_i^o$, $b_i = k_i^o D_i^o$, $L_{ij} = k_{ij}^b D_i^b$. The balance sheets of the banks do not provide information to pin down the parameters k_i^o and k_{ij}^b.[12] The parameters, k_i^o and k_i^b, are a bit easier to estimate from banks' cashflow statements and income statements. The parameters k_{ij}^b are much harder to estimate. We follow the approach used by Upper and Worms (2004).

The assets and liabilities of the twenty two German banks are summarized in Table 1. As the twenty two banks are not the whole banking industry of German, there are other banks, both domestic and foreign, that are part of the German banking industry and interact with the twenty two banks. In Table 1, we added the twenty third bank. This bank can be viewed as the rest of the German banking industry. The asset of the bank is the difference of the total assets of German banks minus the total assets of the twenty two banks. The liability and equity of the bank are obtained similarly. The value in the Claim on Banks is equal to the difference between sum of Liabilities to Banks and total Claims on Banks of the twenty two banks.

We assume that every bank interacts with all the other banks. That is, the banking network connection is complete. Under this assumption, the connection among banks is the strongest. The contagion effect is possibly the strongest for large shocks (Acemoglu, Ozdaglar and Tahbas-Salehi (2013)). We set $k_i^o = 0.04$ and $k^b = 0.04$. These numbers are typical for the banks in the system. Following the approach of Upper and Worms (2004), the estimated liability matrix is given by Table 2. In the table, the banks are sorted by size. The first row and the first column of the table are the list of the banks according to their sizes. The entries of the matrix suggest that the smaller banks, which are mostly spacial banks, do not have strong connections

[12] This is where information from the income and cashflow statements can help. However, as those statements typically do not provide detailed information on the flows of funds at the level of individual banks either, one should not expect the information from those statements would allow us to pin down k_i^b and k_{ij}^b precisely.

Table 1: Assets and Liabilities of Twenty Two German Banks

Bank	Asset	Liability	Equity	Claim on Banks	Liability to Banks	Ext. Asset	Ext. Liability
1	42981	40531	2450	2531	1589	40450	38942
2	34695	32851	1844	1128	8231	33567	24620
3	40521	38360	2161	3029	5020	37492	33340
4	24732	23749	983	2235	10169	22497	13580
5	23437	20246	3191	848	18424	22589	1822
6	70682	68065	2617	20484	23708	50198	44357
7	81932	78738	3194	49440	5550	32492	73188
8	34592	33774	818	3264	4297	31328	29477
9	145350	127467	17883	34287	40727	111063	86740
10	90992	82109	8883	2019	11134	88973	70975
11	22546	21848	698	3091	2653	19455	19195
12	13444	13094	350	1202	3630	12242	9464
13	1580760	1526020	54739	93982	133229	1486780	1392790
14	549661	522725	26936	87545	77694	462116	445031
15	273523	260120	13403	47577	58030	225946	202090
16	386978	372824	14154	74214	91361	312764	281463
17	190307	179136	11171	46424	62367	143883	116769
18	200845	193570	7275	27481	59181	173364	134389
19	109022	104498	4524	5156	18212	103866	86286
20	178083	170996	7087	21396	34106	156687	136890
21	116073	112305	3768	30728	39001	85345	73304
22	51360	49139	2221	21890	26994	29470	22145
23	3341693	3162878	178815	155356	0	3186337	3162878

Table 2: Liability Matrix

	12	11	5	4	8	2	3	1	22	6	7	10	19	21	9	20	17	18	15	16	14	13	23
12	0	1	0	1	1	0	1	1	5	5	11	0	1	7	8	5	12	7	12	19	22	27	35
11	0	0	0	0	1	0	1	0	4	4	8	0	1	5	6	4	8	5	9	14	16	20	26
5	1	4	0	3	4	1	4	3	26	24	58	2	6	37	42	26	59	34	60	98	114	135	180
4	1	2	1	0	2	1	2	2	15	14	32	1	3	21	23	14	32	19	33	54	63	75	99
8	0	1	0	1	0	0	1	1	6	6	13	1	1	9	10	6	14	8	14	23	27	32	42
2	1	2	0	1	2	0	2	1	12	11	26	1	3	17	19	12	26	15	27	44	51	60	80
3	0	1	0	1	1	0	0	1	7	7	16	1	2	10	11	7	16	9	16	27	31	37	49
1	0	0	0	0	0	0	0	0	2	2	5	0	1	3	4	2	5	3	5	9	10	12	16
22	2	5	2	4	6	2	5	4	0	37	87	4	9	56	63	39	88	52	90	148	172	204	271
6	2	5	1	3	5	2	5	4	35	0	76	3	8	49	55	34	77	45	79	130	150	178	237
7	0	1	0	1	1	0	1	1	8	8	0	1	2	12	13	8	19	11	19	32	37	43	58
10	1	2	1	2	2	1	2	2	16	15	35	0	4	23	25	16	35	21	36	60	69	82	109
19	1	4	1	3	4	1	4	3	26	24	57	2	0	37	42	26	58	34	59	98	113	134	179
21	3	8	2	6	8	3	8	6	58	54	127	5	13	0	92	57	129	76	132	217	251	298	396
9	3	8	2	6	9	3	8	7	61	57	133	5	14	87	0	60	136	80	138	228	264	313	416
20	3	7	2	5	7	3	7	6	50	47	110	5	12	71	80	0	111	66	114	187	217	257	342
17	5	13	4	9	14	5	13	11	95	88	208	9	22	135	151	93	0	124	215	356	411	488	649
18	5	12	3	9	13	4	12	10	87	81	192	8	20	125	140	86	195	0	199	328	380	451	599
15	5	12	3	9	13	4	12	10	88	82	194	8	21	126	141	87	197	116	0	331	383	455	605
16	8	20	6	15	21	7	20	16	145	136	320	13	34	208	232	144	325	191	331	0	632	750	997
14	7	17	5	13	18	6	17	14	126	117	277	11	29	180	201	124	282	166	287	473	0	650	864
13	12	30	8	22	32	11	30	25	222	207	488	20	52	317	355	219	496	292	505	834	964	0	1521
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3: Row Sums and Column Sums of the Liability Matrix

Bank	$c_j = \sum_i L_{ij}$	$r_i = \sum_j L_{ij}$	Ratio	$L_{i,23}$
12	60	181	0.33	35
11	155	132	1.17	26
5	41	921	0.04	180
4	114	509	0.22	99
8	164	216	0.76	42
2	54	413	0.13	80
3	155	250	0.62	49
1	128	79	1.62	16
22	1094	1350	0.81	271
6	1026	1183	0.87	237
7	2473	276	8.96	58
10	100	559	0.18	109
19	258	910	0.28	179
21	1535	1949	0.79	396
9	1713	2038	0.84	416
20	1069	1709	0.63	342
17	2320	3118	0.74	649
18	1374	2959	0.46	599
15	2380	2902	0.82	605
16	3710	4571	0.81	997
14	4377	3884	1.13	864
13	4701	6662	0.71	1521
23	7770	0	∞	0

among themselves. They tend to be net payers to the larger banks, which are mostly Landesbanken or commercial banks. The large banks have strong connections among themselves.

The row sums and column sums are given in Table 3. Element j of the row sums is the sum of the elements of the jth row of the liability matrix. Similarly, element i of the column sums is the sum of the elements of the ith column of the liability matrix. When the row sum of bank j is greater than its column sum, bank j's liability to other banks is greater than its receivables from other banks. For example, bank 5's liability to other banks is 921 whereas its receivables from other banks in the system is 41. Banks that pays less than it receives from other banks in the system is a net receiver. Otherwise it is a net payer. The smaller the ratio of column sum to row sum of a bank, the less vulnerable the bank is to contagion effect because a relatively smaller fraction of its liability is paid from its receivables from other banks. The ratios for the banks in the system varies from as low as 0.04 for bank 5 to as high as 8.96 for bank 7. In the case of bank 5, the low ratio means that even if all the other banks in the system stop making payments to it, it would affect at worst 4% of its liability. It relies mainly on its own assets to pay for its liabilities to other banks. In that sense, it is not significantly vulnerable to contagion effect. If it fails, it would be due to its own weakness. On the other hand, a high ratio implies the bank is more vulnerable to contagion effect. For example in the case of bank 7, the ratio is 8.96. It is a net receiver. Potentially, the bank can count on the payments it receives from other banks to pay for its liabilities to other banks. If that is the case, it could be more vulnerable to contagion effect.

Table 4 reports the P-matrix of the 22 banks.[13] To make the matrix more readable, we have set p_{ij} to zero if the actual number is less than 0.005. The table reveals several interesting features of the system. For example, the first eight columns of the table are approximately zero. It shows that smaller banks do not receive much from the system. They typically are net payers of the system. Also, the upper 8 by 8 sub-matrix is approximately zero. It suggests that the small banks do not pay each other. The sub-system of the eight smallest banks is a system of 8 isolated banks. The smaller banks, however, do make payments to larger banks. Larger banks typically receive payments from all other banks in the system. The behavior of mid-sized banks varies. Some receive payments from all banks, while others such as banks 10 and 19 receive approximately zero payments from other banks. Large banks connect with each other. The last column provides the sum, $\sum_{j=1}^{23} p_{ij}$. It is the fraction of the total payment of bank i to the rest of the banks in the system. For bank 5, the number is 0.93, which suggests that 93% of the payments made by bank 5 go to other banks in the system. For bank 1, the number is 0.05, which suggests that 95% of the payments made by bank 1 are to liabilities outside the system. On average, 29% of the payments made by the banks in the system are to the banks in the system. The standard deviation is 20%. Tables 2 and 4 together suggest that there are significant connections among the banks, although the strength of the connection varies significantly across banks.

6.2 Default Sequence and Financial Crisis

In this section, we examine how adverse shocks to banks affect the default sequence of the banks in the system.

We start with the case where k_a is equal to k_b which are both equal to 0.04. As described earlier, these numbers are typical for banks in the system. This case serves as the benchmark that shows under normal conditions how the system of the twenty two German banks evolve over time. Table 5 shows the banks that will eventually default. Table 5 tells us several things about the system of the twenty two German banks. Firstly, the first column tells us which banks will eventually default. The last column shows that the equity-net-cashflow ratios of the four banks are all negative. Thus these four banks are fundamentally weak. In other words, they fail not because of contagion but because of the weakness in the fundamentals of the banks. The smallest of the four ratios in absolute value is that for bank 4. Thus bank 4 defaults first, according to the prediction of our theory, and the time of default is equal to the absolute value of the ratio. The ratios of the other banks, which are not reported here, are all positive. Next, since all the default times of the four banks are relatively far away, the implication is that for the near future, the bank system is not subject to financial crisis. Consequently the systemic risk of the twenty-two German banks, or more generally, the German banking system, is minimum. Lastly, the default times are far apart from each other. The first to default bank is bank 4, which defaults in about 24.6 years from now, followed by banks 12, 5

[13] To estimate the liability matrix we need to introduce another bank into the system to make the row sum equal to column sum. So the matrix is a 23 by 23 matrix. We report the 22 by 22 P-matrix of the original 22 banks.

Table 4: P Matrix

	12	11	5	4	8	2	3	1	22	6	7	10	19	21	9	20	17	18	15	16	14	13	23	sum
12	0	0	0	0	0	0	0	0	0.01	0.01	0.02	0	0	0.01	0.01	0.01	0.02	0.01	0.02	0.03	0.04	0.05	0.06	0.32
11	0	0	0	0	0	0	0	0	0	0	0.01	0	0	0.01	0.01	0	0.01	0.01	0.01	0.02	0.02	0.02	0.03	0.15
5	0	0	0	0	0	0	0	0	0.03	0.02	0.06	0	0.01	0.04	0.04	0.03	0.06	0.03	0.06	0.10	0.11	0.14	0.18	0.93
4	0	0	0	0	0	0	0	0	0.01	0.01	0.03	0	0	0.02	0.02	0.01	0.03	0.02	0.03	0.05	0.06	0.07	0.09	0.48
8	0	0	0	0	0	0	0	0	0	0	0.01	0	0	0.01	0.01	0	0.01	0.01	0.01	0.02	0.02	0.02	0.03	0.15
2	0	0	0	0	0	0	0	0	0.01	0.01	0.02	0	0	0.01	0.01	0.01	0.02	0.01	0.02	0.03	0.04	0.04	0.06	0.29
3	0	0	0	0	0	0	0	0	0	0	0.01	0	0	0.01	0.01	0	0.01	0.01	0.01	0.02	0.02	0.02	0.03	0.16
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.01	0.01	0.01	0.01	0.05
22	0	0	0	0	0	0	0	0	0	0.02	0.04	0	0	0.03	0.03	0.02	0.04	0.02	0.04	0.07	0.08	0.09	0.12	0.60
6	0	0	0	0	0	0	0	0	0.01	0	0.03	0	0	0.02	0.02	0.01	0.03	0.02	0.03	0.04	0.05	0.06	0.08	0.40
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.01	0	0.01	0.01	0.01	0.01	0.02	0.09
10	0	0	0	0	0	0	0	0	0	0	0.01	0	0	0.01	0.01	0	0.01	0.01	0.01	0.02	0.02	0.02	0.03	0.16
19	0	0	0	0	0	0	0	0	0.01	0.01	0.01	0	0	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.03	0.03	0.04	0.21
21	0	0	0	0	0	0	0	0	0.01	0.01	0.03	0	0	0	0.02	0.01	0.03	0.02	0.03	0.04	0.05	0.06	0.08	0.40
9	0	0	0	0	0	0	0	0	0.01	0.01	0.02	0	0	0.02	0	0.01	0.02	0.01	0.03	0.04	0.05	0.06	0.08	0.37
20	0	0	0	0	0	0	0	0	0.01	0.01	0.02	0	0	0.01	0.01	0	0.02	0.01	0.02	0.03	0.03	0.04	0.05	0.24
17	0	0	0	0	0	0	0	0	0.01	0.01	0.03	0	0	0.02	0.02	0.01	0	0.02	0.03	0.05	0.05	0.06	0.08	0.40
18	0	0	0	0	0	0	0	0	0.01	0.01	0.02	0	0	0.01	0.02	0.01	0.02	0	0.02	0.04	0.05	0.05	0.07	0.36
15	0	0	0	0	0	0	0	0	0.01	0.01	0.02	0	0	0.01	0.01	0.01	0.02	0.01	0	0.03	0.03	0.04	0.06	0.26
16	0	0	0	0	0	0	0	0	0.01	0.01	0.02	0	0	0.01	0.01	0.01	0.02	0.01	0.02	0	0.04	0.05	0.06	0.29
14	0	0	0	0	0	0	0	0	0.01	0.01	0.01	0	0	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0	0.03	0.04	0.18
13	0	0	0	0	0	0	0	0	0	0	0.01	0	0	0.01	0.01	0	0.01	0	0.01	0.01	0.02	0	0.02	0.11
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00

Table 5: Default Sequence

Bank	Default Time (years)	Equity/Net-Cashflow
4	24.6	-24.6
12	34	-34.0
5	66.3	-66.3
18	259.7	-279.8

and 18, defaulting in 34, 66.3 and 259.7 years, respectively. Clearly there is no clustering of bank failures.

Next, we will subject each bank to an adverse shock and examine possibility of contagion. Table 6 shows effect of 50% negative shock to exogenous cash inflow, α_j, of bank j, $j = 1, \ldots, 23$, on banks in the system. There are 23 panels in the table. Each panel describes the effect of the shock to one bank. The order of the panels is organized according to the size of the banks. For example, the first panel in the top row describes the effect of the shock of bank 12. The second panel in the top row describes the effect of the shock of bank 11, and so on. The symbol ∞ in the table is used to indicate that the number of years is greater than or equal to 100. For example, in panel one, the default time of bank 18 is greater than 100 years.

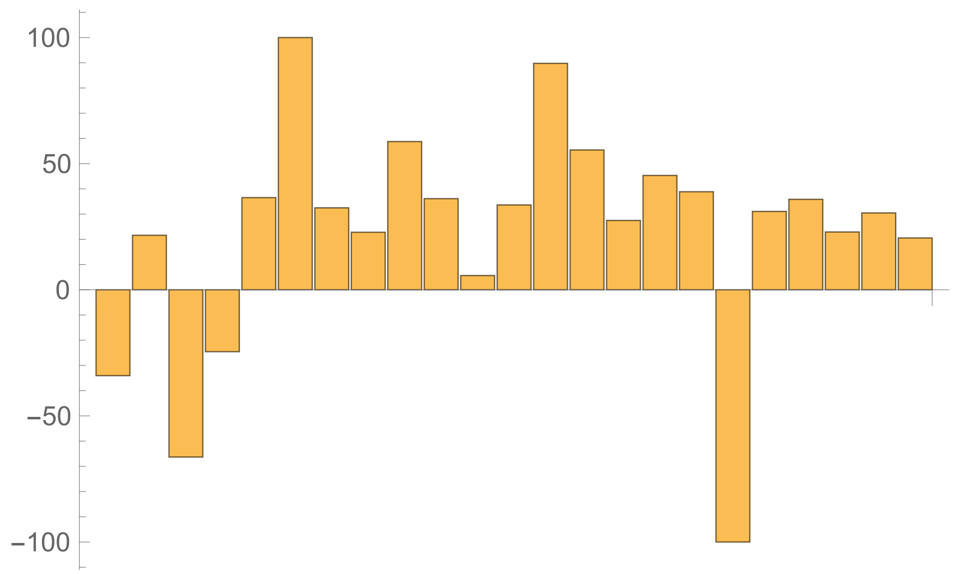

Figure 3: Equity/net-cashflow Ratios.

The first thing to notice in Table 6 is that banks, 4, 5, 12 and 18 are in every panel. This is no surprise as they are fundamentally weak. The twenty-three panels in the table tells us quite a bit about the system of these banks. As bank 23 does not have liability to any of the other twenty two banks, it will never drag any of the other twenty two banks into default. As explained earlier, bank 23 can be viewed as the aggregation of all the banks outside the system of the twenty two banks. Then if there is any default in the system, it can only be due to the fundamental weakness of the banks inside the system or due to the contagion effect generated inside the system. This is shown in the last panel in Table 6, in which only bank 23 plus banks,

Table 6: Default Sequence When Banks Experience Individual Shocks

Bank	Time	Bank	Time	Bank	Time	Bank	Time	Bank	Time	Bank	Time	Bank	Time	Bank	Time
12	1.4	11	2.0	5	6.4	4	2.0	8	1.4	2	2.8	3	3.2	1	3.5
4	24.4	4	24.5	4	24.0	12	33.0	4	24.4	4	24.3	4	24.4	4	24.5
5	66.2	12	33.8	12	32.3	5	66.0	12	33.6	12	33.1	12	33.5	12	33.9
18	∞	5	66.2	18	∞	18	∞	5	66.2	5	66.0	5	66.1	5	66.2
		18	∞					18	∞	18	∞	18	∞	18	∞

Bank	Time	Bank	Time	Bank	Time	Bank	Time	Bank	Time	Bank	Time	Bank	Time	Bank	Time
22	4.0	6	2.8	4	24.6	10	5.9	19	2.2	21	2.3	9	11.4	20	2.4
4	24.1	4	24.0	12	34.0	4	24.2	4	23.9	4	23.5	4	24.0	4	23.5
12	32.6	12	32.4	7	38.2	12	33.0	12	32.1	12	31.1	12	32.2	12	31.0
5	65.8	5	65.7	5	66.3	5	66.0	5	65.7	5	65.3	5	65.5	5	65.3
18	∞	18	∞	18	∞	18	∞	18	∞	18	∞	18	∞	18	∞

Bank	Time	Bank	Time	Bank	Time	Bank	Time	Bank	Time	Bank	Time	Bank	Time		
17	4.3	18	2.1	15	3.3	16	2.4	14	3.3	13	2.0	23	3.2		
4	23.1	4	22.7	4	23.0	4	21.9	4	22.3	4	20.1	4	24.6		
12	29.9	12	28.9	12	29.6	12	27.1	12	28.0	12	23.1	12	34.0		
5	64.8	5	64.5	5	64.7	5	63.7	5	64.0	22	38.9	5	66.3		
18	∞	22	∞	18	∞	18	76.2	18	84.3	18	47.8	18	∞		
						22	∞	22	∞	21	53.6				
						21	∞	21	∞	5	61.5				
						2	∞			6	∞				
										2	∞				
										16	∞				

Table 7: Contagion Effect

Bank	13	16	14	22	4	12	21	6	18	5	17	8	20	15	2	19
Time	2.0	2.4	3.2	15.7	16.9	17.5	19.1	24.4	26	57	60	90	116	174	215	1422

4, 5, 12 and 18 will eventually default. Another interesting common pattern in the panels is that when the adverse shock hit a bank, only that bank plus the four weak will default. This is true of 19 out the 23 panels. Moreover, in these 19 panels, when the bank hit by the shock is not among the four weak banks, the failure of the hit bank has very little effect on the default times of the four weak banks. This suggests that in the majority cases, while there are connections among banks, most banks are effectively isolated as far as contagion effect on default is concerned. This is largely because the equity/net-cashflow ratios are mostly significantly positive except for the four weak banks as shown in Figure 3. The most important character of the system in Table 6 is of course the contagion effect. The effect is exhibited in panels 18, 20, 21 and 22. In panel 22, there are ten banks that will eventually fail. In addition to banks 13, 4, 5, 12 and 18, banks 2, 6, 16, 21 and 22 will also fail. These later five banks start with positive equity/net-cashflow ratios, meaning they are net cash receivers to begin with, but eventually are dragged into default by the failure of bank 13. Finally, there is no clustering of banks failure in any of the 23 panels. That is, there is no financial crisis.

Next we look at the case where banks 13, 14 and 16 are hit by adverse shocks at the same time. Table 7 shows that altogether 16 banks will fail in the end. The contagion effect arising from shocks to banks 13, 14 and 16 will lead to the failure of 9 banks that are otherwise healthy. However, again there is no clustering of bank failures, except for the three banks that are hit by adverse shocks.

If we shock all the banks, this is what we have. In two years, 12 banks will fail with 10 of them happening between year 1 and 2, and in 3 years, 19 banks will fail from year 1 to 3. Even though all banks will fail,

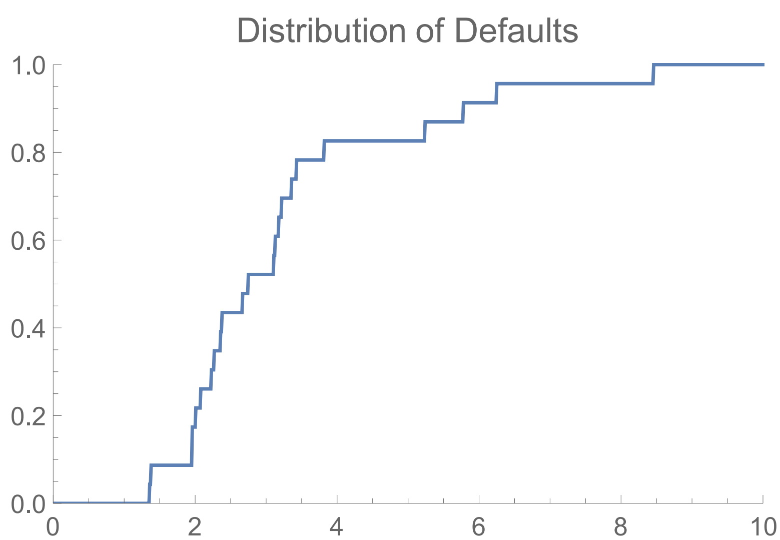

Figure 4: Equity/net-cashflow Ratios.

the first default will not happen until year 1.35.

Finally, we look at systemic risk. We shock the k_i^o for all banks at the same time. We repeat the shocks for 5000 times and then calculate the statistical likelihood of bank defaults. Specifically, let $k_i^o = 0.04exp\left(\eta_i\right)$,

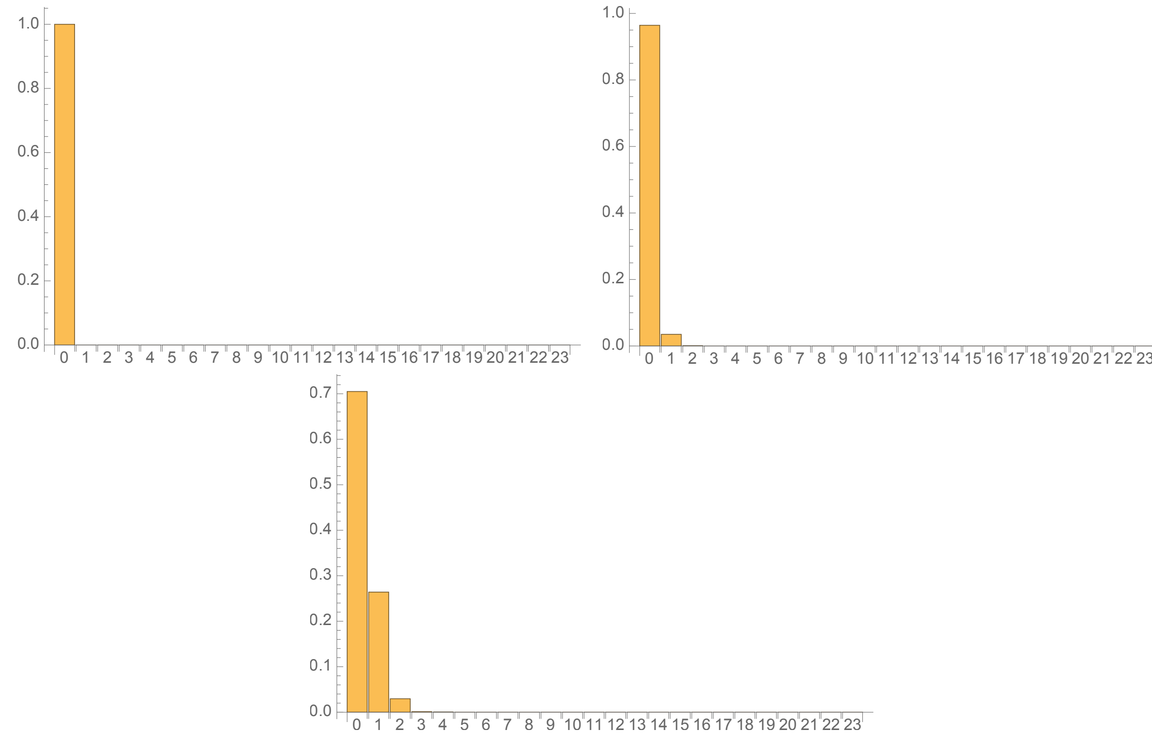

Figure 5: Systemic Risk. $\eta = 1$, 2 and 3 years. Correlation coefficient is zero. Standard deviation is 0.2

$i = 1, 2, \ldots, 23$, where $(\eta_1, \eta_2, \ldots, \eta_{23})$ follows the jointly normal distribution, $N(0, \Sigma)$. The variance-covariance matrix is such that the standard deviation ξ_i of all ξ_i are the same. We set the common value $\sigma = \sigma_i$ to 0.2. By setting such value for the common standard deviation, we are assuming that an adverse one standard deviation shock the bank i will cause k_i^o to drop from 0.04 to 0.327, about 18% drop, which corresponds to 18% drop in α_0. That is, the exogenous cash inflow of the bank drops by 18%. A two standard deviation adverse shock leads to 33% decrease of k_i^0. That is, 33% drop in exogenous cash inflow. The shocks ξ_i can be correlated. If the correlations of ξ_i are all equal to zero, the shocks are all idiosyncratic shock. If the shock are correlated, then the shocks have a systematic component. We look at two cases: (a) the correlation coefficients are all equal to zero, and (b) the correlation coefficients are all equal to 0.7. We shock the system of twenty three banks 5000 times. We then calculate there are 0, 1, 2, ..., 23 banks that will default in the next one, two and three years. The results are shown in Figure 5 and 6. In both cases, the probability of having more than one bank to default in the next one year is zero. The probability of having more than one bank to default in the next two year is negligible, 0.0006. When the correlation coefficient is 0.7, the probability is 0.0046. If the horizon is three years, the probabilities of having more than three banks are 0.0002 and 0.017, respectively.

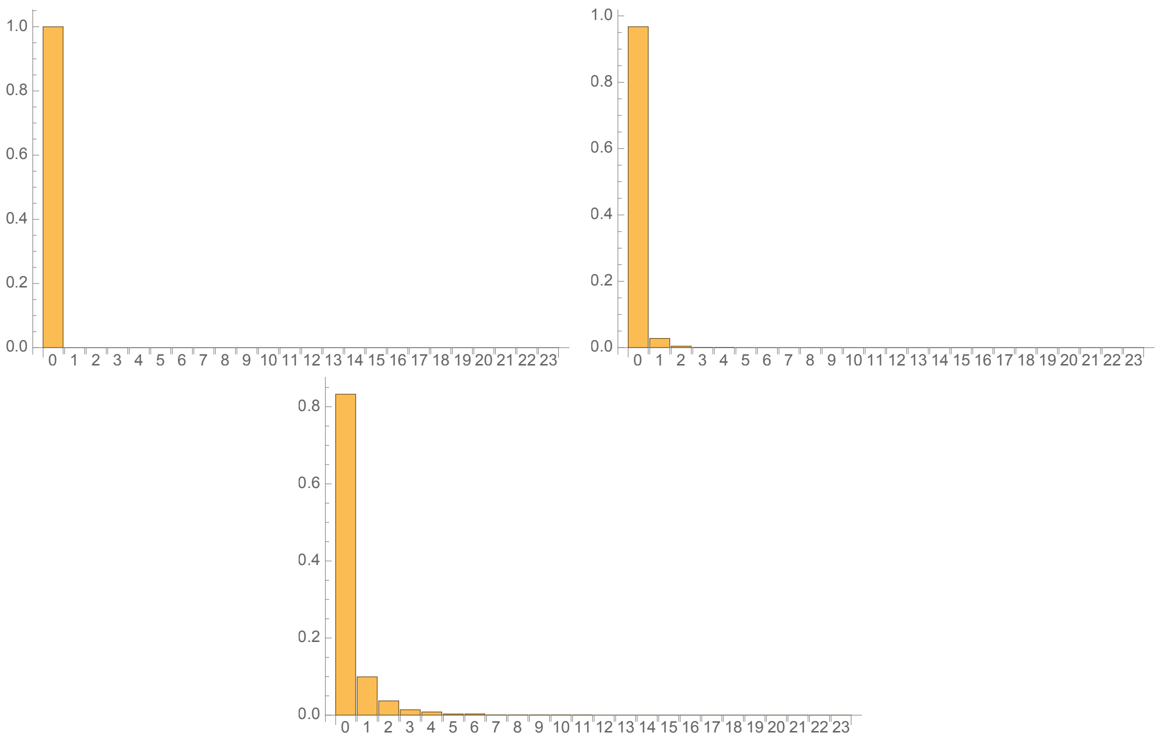

Figure 6: Systemic Risk. $\eta = 1$, 2 and 3 years. Correlation coefficient is 0.7. Standard deviation is 0.2

For the regulator who monitors the systemic risk of the banking system, these numbers suggest that the occurrence of systemic risk in the next one year is a zero probability event. To use our terminology, suppose that a $(\eta, q) = (1, 0.1)$ financial crisis is defined as when there is more than 10% of the banks that will default in the next year. Then the probability of a $(1, 0.1)$ financial crisis is zero whether the correlation coefficients are zero or 0.7. Even if we extend the horizon to $\eta = 3$ years, then the probability is 0.14% when the shocks are all idiosyncratic, while the probability is 3.08% when the correlation coefficient of the shocks is 0.7.

7 Conclusion

It has been well argued that network externality can give rise to financial contagion. Network may arise for a variety of reasons such as diversification and insurance need (Allen and Gale (2000), Freixas, Parigi, and Rochet (2000) and Leitner (2005)). Once the network is generated, the contagion effect can be affected by a number of factors such as lack of monitoring (Rochet and Tirole (1996)), the structure of the network (Acemoglu, Ozdaglar, and Tahbaz-Salehi (2013)), and the illiquidity of bank assets (Cifuentes, Ferrucci, and Shin (2005)).

However, the factors and mechnism that determine whether a crisis will occur have not been as well understood. The existing literature is largely based on static network models. A static model is best suited for capturing the interconnection aspect of a network. For example, it can be used to show how the structure of the network may affect the likelihood of contagion (Acemoglu, Ozdaglar, and Tahbaz-Salehi (2013)). While the understanding of the role of interconnectedness in the occurrence of financial crisis is important, it is not the full story. The contrast of stock vs flow, for example, is missing in static models. As a result, there is no meaningful distinction between bank capital and cash flow. To capture the distinction, a dynamic model is needed. In this paper, we introduce dynamics into an otherwise standard network model of financial system, i.e., Eisenberg-Noe model (Eisenberg and Noe (2001)). The dynamic model developed allows us to study the dynamics of bank defaults and the financial crises that can arise due to contagion effect. An important insight from the study is that while the possibility of contagion is determined by interconnectedness of the financial network, which is largely determined by the flows among the banks, whether a financial crisis may occur depends on the capitals of the banks in the system. The amount of capital held by the banks determines how long a bank can remain solvent. The profile of bank capitals and net cashflows to banks together determine the default times of the banks. A financial crisis occurs when a large number of banks default in a short span of time. Thus, it is the profile of initial capitals and net cashflows of the banks that pre-determines the occurrence of a financial crisis.

Drawing on that insight, we develop an index that predicts the occurrence of a financial crisis. Then an intuitive measure of systemic risk is provided. To illustrate the potential usefulness of our model, we conduct an analysis of the system of twenty two German banks. We show how many of the banks are fundamentally weak, where the contagion effect may arise from and how large the systemic risk is.

Our study contributes to a better understanding of the factors that give rise to financial crises and systemic risk. It should also be of interest not only to the academic profession but also to those practitioners whose responsibility is to monitor the systemic risk of the financial system and to maintain the well functioning of the financial markets.

A Proof of Lemmas and Propositions

Proof of Proposition 1: (i) Since P is substochastic with spectral radius < 1, $P^n \to 0$ as $n \to \infty$; hence, the fixed-point iteration is a contraction (when n is sufficiently large), which guarantees convergence. [In fact, the mapping is increasing in λ (since P is non-negative). Thus, if we start with $\lambda = \alpha$, the sequence (of iterative solutions) will converge monotonically to the solution.]

(ii) Taking minimum against ℓ on both sides of (26), we have

$$\ell \wedge \lambda = \ell \wedge \big[\alpha + (\lambda \wedge \ell)P\big],$$

from which we obtain $\pi = \ell \wedge \lambda$ as a solution to (5).

(iii) From (26), we have

$$\lambda = \alpha + (\lambda \wedge \ell)P \leq \alpha + \ell P.$$

In addition, from (**??**), we know $\theta_i < 0$ means $\alpha_i + (\ell P)_i < \ell_i$, and hence implies $\lambda_i < \ell_i$. $\square$

Proof of Proposition 4: Bank i is the first to default and hence $\tau_i = \tau^{(1)} = t_0$. The time of first default is in $[t_0, t_0 + \eta)$. By (34),

$$z_j(\tau^{(k+1)}) = z_j(\tau^k) + \theta_j^{(k)}(\tau^{(k+1)} - \tau^{(k)});$$

Now suppose that bank j is the Kth to default so that $\tau_j = \tau^{(K)}$. Since $\theta_j < 0$ for $j \in \mathcal{D}'$, we have

$$\tau_j = \sum_{k=0}^{K-1} \tau^{(k+1)} - \tau^{(k)} = \sum_{k=0}^{K-1} \frac{z_j(\tau^{(k+1)}) - z_j(\tau^k)}{\theta_j^{(k)}}$$

where $\tau^{(0)} = 0$. By (33), $\theta_j^{(k+1)} < \theta_j^{(k)}$. Note also $z_j(\tau^{(k+1)}) - z_j(\tau^k) < 0$. Thus,

$$\frac{z_j(\tau^{(k+1)}) - z_j(\tau^k)}{\theta_j^{(k)}} + \frac{z_j(\tau^{(k)}) - z_j(\tau^{k-1})}{\theta_j^{(k-1)}} \leq \frac{z_j(\tau^{(k-1)}) - z_j(\tau^{k+1})}{-\theta_j^{(k-1)}}$$

Hence

$$\tau_j = \sum_{k=0}^{K-1} \frac{z_j(\tau^{(k+1)}) - z_j(\tau^k)}{\theta_j^{(k)}} \leq \frac{z_j(\tau^{(0)}) - z_j(\tau^K)}{-\theta_j^{(0)}} = \frac{z_j(0)}{-\theta_j}$$

where the last equality follow from $z_j(\tau^K) = 0$ as bank j is the Kth to default. Subtracting $\tau^{(1)}$ from both sides yields,

$$|\tau_j - \tau^{(1)}| = \left|\frac{z_j(0)}{-\theta_j} - \frac{z_i(0)}{-\theta_i}\right| \leq \eta.$$

Thus $\tau_j \in [t_0, t_0 + \eta)$ as claimed. $\square$

Proof of Proposition 5: As shown in Proposition 4, for $j \in \mathcal{D}_1$, $\tau_j \in [t_0, t_0 + \eta]$. Now let $j \in \mathcal{D}_2$. Suppose that $p_{kj} > 0$ for some $k \in \mathcal{D}_1$. Suppose that bank k is the mth to default. Then after bank k defaults, the payment from bank k to bank j will be strictly less than $\ell_k p_{kj}$. As a result, by (33), $\theta_j^{(m)} < \theta_j = 0$. As bank j has zero initial bank capital, $z_j(0) = 0$, and because $\theta_j^{(k)}(t) \leq \theta_j$ for all $k \leq m$ its bank capital does not increase over time, $z^{(k)}(t) \leq 0$ for all $k \leq m$. Thus, $\tau_j \leq \tau_k$ and hence $\tau_j \in [t_0, t_0 + \eta]$ as claimed. $\square$

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