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Note

Prospect theory and liquidation decisions

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Abstract

We solve a liquidation problem for an agent with preferences consistent with the prospect theory of Kahneman and Tversky [Econometrica 47 (1979) 263–291]. We find that the agent is willing to hold a risky project with a relatively inferior Sharpe ratio if the project is currently making losses, and intends to liquidate it when it breaks even. On the other hand, the agent may liquidate a project with a relatively superior Sharpe ratio if its current profits rise or drop to the break-even point. Our results capture the spirit of the disposition effect and the break-even effect documented in empirical and experimental studies.

JEL classification: G39; G19; D81

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1. Introduction

The prospect theory of Kahneman and Tversky [16] represents an alternative theory of decision making under uncertainty, as opposed to the standard Morgenstern–von Neumann utility theory. Based on a series of experimental observations, Kahneman and Tversky propose a value function defined on the gains or losses relative to a reference point, instead of the absolute level of consumption or wealth. Specifically, they state that “the value function is (i) defined on deviations from the reference point; (ii) generally concave for

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gains and commonly convex for losses; (iii) steeper for losses than gains.'' Prospect theory has been widely supported by experimental studies of human behavior.

Recently, prospect theory has been employed to explain a series of empirical and experimental findings on the liquidation decisions of economic agents. The literature does not, however, provide a formal model to analyze the liquidation decisions under prospect theory. One of the difficulties lies in the non-smoothness of the value function, which may cause failure of the standard first-order conditions in the dynamic optimization problem.

The objective of this paper is to investigate formally the effect of prospect theory on the liquidation decisions of economic agents. Specifically, we analyze an agent with a project (or an indivisible asset). The agent can either wait for the project to pay off by itself, an event which arrives exogenously according to a Poisson process, or choose to liquidate the project before its natural payoff. In either case, the agent gets a payoff equal to the fundamental value of the project, which fluctuates according to a Brownian motion with constant drift and volatility. According to a benchmark model in which the agent has a standard exponential utility function, the agent chooses either to liquidate the project immediately if the risk-adjusted growth rate (''Sharpe ratio'') of the project is not high enough for the agent's risk aversion, or to hold the project until it pays off by itself if its Sharpe ratio is high enough.

In order to capture the preferences of prospect theory, we specify a two-piece exponential direct value function for the agent. The agent only realizes his direct value function when the project is liquidated or when the project pays off by itself. The reference point is chosen to be the break-even point, which is consistent with the status quo used in the literature. Above the reference point, the value function is a concave exponential function of gains. Below the reference point, it is a convex exponential function of losses. In addition, some parameter restrictions are imposed to ensure that there is a concave kink in the value function around the reference point. This makes the agent more sensitive to losses than to gains.

Since the agent's direct value function is determined by the gains and losses from the project, his risk aversion varies with the paper gains and losses of the project through his expected value function. This change in risk aversion drives the agent's liquidation decisions in response to the paper gains and losses in the project. Specifically, there are two forces in play. On the one hand, the convexity in the agent's direct value function of losses causes the agent to delay realization of losses. On the other hand, the loss aversion (higher sensitivity to losses than to gains around the reference point) induces the agent to liquidate around the break-even point. The net effect of these two forces can cause the agent to delay liquidation of inferior projects as well as to accelerate liquidation of superior projects, depending on the range of the project's Sharpe ratio and its current gains or losses.

We discuss these effects based on an exogenously specified Sharpe ratio and initial gains or losses of the project. We can intuitively think of many different ways in which this problem may arise. One possibility is that the project's fundamental is not observed when the project is initiated, but is observed sometime later, after fundamentals have changed. Another possibility is that the agent is only allowed to liquidate the project after certain period of time, during which the fundamentals may change. Still another possibility is that when the agent invests in the project, the project's Sharpe ratio is sufficiently high. After a certain period of time, an external shock causes the Sharpe ratio to increase or decrease. Now the project is already in gains or losses and the agent is faced with a liquidation decision.

We are able to classify four scenarios according to the project's Sharpe ratio. In scenario I, the Sharpe ratio is very high (exact conditions are given in Section 4). In this case, the agent will always hold the project until it pays off by itself, because the Sharpe ratio of the project is high enough to compensate the agent for his loss aversion and risk aversion.

In scenario II, the Sharpe ratio is relatively high in the sense that an agent with the standard exponential utility would choose to hold the project, [1] but the Sharpe ratio is not high enough to compensate for the agent's loss aversion. In this case, we find that the agent would always liquidate the project at the break-even point. Intuitively, if the project is currently in paper losses, the convexity in the agent's direct value function would induce the agent to hold the project at least until it breaks even. Once the project breaks even, the kink in the direct value function at this point (due to the loss aversion, i.e., the higher sensitivity to losses than to gains) makes the agent very risk averse, and therefore he liquidates the project exactly at the break-even point. If the project currently registers paper gains, the loss aversion will also induce the agent to liquidate exactly at the break-even point.

Although the strategy to liquidate at the break-even point is a typical threshold strategy that has been analyzed in most optimal stopping problems such as exercising American options and real options, [2] we cannot solve the optimal threshold through the standard smooth pasting conditions [3] due to the presence of the kink in the agent's value function. Instead, we find the break-even point as a corner solution. We are also able to provide a verification theorem using the Tanaka formula (a generalized Ito rule for non-differentiable functions) to show that liquidating at the break-even point is indeed the optimal strategy.

In scenario III, the Sharpe ratio is relatively inferior in the sense that it is not high enough to compensate an agent with a standard utility function to hold the project, and it is not low enough for an agent with a convex preference to losses to liquidate it immediately (the exact conditions are given in Section 4). In this case, if the project is currently in paper gains, the agent will liquidate it immediately because its fundamental is inferior. However, if the project is currently in paper losses, the convexity in the agent's direct value function for losses causes him to delay the liquidation until the project breaks even. Once again, the smooth pasting conditions fail in this case, and liquidating at the break-even point is a corner solution.

In scenario IV, the Sharpe ratio is so inferior that the agent would always liquidate the project immediately whether it is currently in paper gains or losses or at the reference point.

In summary, our model shows that prospect theory preferences induce the agent to delay liquidation of a relatively inferior project if it is in losses and to accelerate liquidation of a relatively superior project if it is in gains. These results capture the spirit of the disposition effects labelled by Shefrin and Statman [24]. Shefrin and Statman [24], Odean [21], and Grinblatt and Keloharja [13] find that investors have a tendency to hold losing stocks too long in financial markets. Genesove and Mayer [11] study seller behavior in the housing market using data from downtown Boston. Housing fits our model well in that a house is an indivisible asset which cannot easily be partially liquidated. They find evidence that

[1] We assume that the agent's risk aversion coefficient above the reference point in the prospect theory preference is the same as in the standard exponential utility function.

[2] See, e.g., [8,20] for early developments of real options.

[3] See, e.g., [10] for a review of the smooth pasting conditions.

sellers, who have incurred losses, are reluctant to realize them. Shefrin [23] presents the case of Sony Corporation in which a co-founder led a project. The project had suffered heavy losses, but the co-founder continued to invest in it and refused to accept a sure loss. [4]

In addition, our result that the agent liquidates the project at exactly the break-even point captures the spirit of the break-even effect of Thaler and Johnson [26]. Through real money experiments, Thaler and Johnson find that in the presence of prior losses, gambles that offer a chance to break even are especially attractive. In other words, subjects who suffer initial losses are more willing to take gambles that would allow them to break even.

Technically, our paper solves an optimal stopping problem with a non-smooth objective function, using the dynamic programming approach, but without relying on the smooth pasting conditions.

The existing theoretical literature on prospect theory has focused on the portfolio choice and asset pricing, [5] but to our knowledge it has not solved a liquidation problem. As a result, the disposition effect has not been modelled in a rigorous manner and the break-even effect has not been explained.

The rest of the paper is organized as follows. Section 2 describes the model structure. Section 3 provides a benchmark model with standard exponential utility functions. Section 4 analyzes the liquidation decisions under prospect theory. Section 5 offers further discussions of the model. Section 6 concludes the paper, and the appendix contains technical proofs.

2. The model

We consider the liquidation decision of an agent who owns a project that requires an initial cost K_0 to start. In the following periods, it takes an additional cost of

$$dK_t = c\,dt \tag{1}$$

to keep the project going. The project pays off only once, and the payoff comes either when the owner decides to terminate the project or when an exogenous force, which arrives according to a Poisson process with a arrival rate λ, causes the project to terminate. When the project is terminated, the payoff equals the value of a fundamental variable, X, at that time. This fundamental variable evolves according to a Brownian motion plus a drift term:

$$dX_t = \mu\,dt + \sigma\,dZ_t, \tag{2}$$

where μ and σ are constant parameters, and Z_t denotes a standard Brownian motion process independent of the Poisson arrival process of the payoff.

We denote the natural arrival time of the payoff by a stopping time τ_1, and the liquidation time chosen by the agent by another stopping time τ_2. The project ends at time

$$\tau = \min\{\tau_1, \tau_2\}, \tag{3}$$

[4] See also [15] on the stock option exercise decisions by employees, and [19,22] on IPOs and American options. Barberis and Thaler [5] provide a detailed review on these studies.

[5] See, for example, [6,3,2,7,12,4]. Other related models include [14,1,9].

and the profit or loss on the project is only realized when the project naturally pays off or when it is liquidated. We denote the profit or loss as the difference between the payoff and the total cost incurred:

$$Y_{t+\tau} = X_{t+\tau} - K_{t+\tau}. \tag{4}$$

The inclusion of the cost is for generality. It shall be seen that all our results would still hold even if we ignore the cost K_t.[6] For example, as a special case we can set $c = 0$ and K_0 (the initial value of K_t in Eq. (1)) to be the initial investment for the project. In this case, the total cost would just be the initial investment. We can also interpret the cost as the total interest cost per period for the firm's capital. Then, K_t represents the total interest proceeds plus the initial investment. Note that Eq. (4) defines the net payoff from the project (the total payoff from the project less the proceeds the manager would otherwise obtain from a risk-free account). In other words, Eqs. (1), (2), and (4) capture the notion of comparing the gains and losses from investing in a risky project to the gains from investing in a risk-free asset, with c being the total interest amount per period.[7]

Although we phrase the liquidation problem in the context of a project, one might also think of the project as a housing position or an indivisible financial asset. The agent borrowed K_0 to acquire the asset, and the holding (or interest) cost is c per period. The agent may be forced to sell the asset at a future stopping time τ_1 for liquidity reasons, or he can choose to sell the asset voluntarily at τ_2. The price of the asset follows the process specified in Eq. (2). To focus on the liquidation decisions, we require that all the asset be sold together, and the gain or loss is realized as given in Eq. (4).

Upon liquidation of the project (or asset), the agent realizes his value function of the gain or loss from the project. For tractability, we assume that the agent has a two-piece exponential direct value function at the time when the project is liquidated:

$$u_{t+\tau}(Y) = \begin{cases} \phi_1 \left(1 - e^{-\gamma_1 Y}\right) & \text{if } Y \geqslant 0 \\ \phi_2 \left(e^{\gamma_2 Y} - 1\right) & \text{if } Y < 0, \end{cases} \tag{5}$$

where $\phi_1 > 0,\ \gamma_1 > 0,\ \phi_2 > 0,\ \gamma_2 > 0$. According to this specification, $Y = 0$ represents the reference point, or equivalently the agent breaks even at this point. Using the break-even point as the reference point is consistent with the status quo proposed by Shefrin and Statman [24] and Odean [21]. Above the reference point, the agent's value function is a concave exponential function, with γ_1 measuring the local absolute risk aversion in this region. Below the reference point, the value function is a convex exponential function, with γ_2 measuring the local absolute risk loving level. In addition, we assume $\phi_1\gamma_1 < \phi_2\gamma_2$ to insure that the agent is more sensitive to losses than to gains around the reference point $[u'(0-) > u'(0+)]$. With the convexity below the reference point, the sensitivity to losses diminishes when losses become large. Thus this value function captures the main features of prospect theory.

The kink in the prospect theory utility function makes the agent's dynamic maximization problem extremely difficult to solve. This is perhaps the reason that the prior literature has

[6] Alternatively, we may assume that the X_t in Eq. (2) denotes the net profit of the project.

[7] We thank a referee for this insight.

only used two straight lines to approximate the prospect theory utility function. In some sense, our specification of the two exponential functions represents an extension of the literature because it captures agent's risk aversion in the positive region as well as his risk seeking behavior in the negative region. It would be ideal to employ the power functions as suggested by Tversky and Kahneman [27], but the agent's problem becomes intractable to solve in this case. Nevertheless, we believe that our results based on the exponential functions are still robust. [8]

The agent maximizes his expected future value function by choosing an optimal liquidation time τ_2:

$$V(Y_t) = \max_{\tau_2} E_t\left[u(Y_{t+\tau})\right]. \tag{6}$$

For simplicity, we have let the time discount rate in the value function be zero. [9] Y_t is the only state variable in this problem, and it measures the profits or losses that could be realized if the agent liquidates the project immediately. We can rewrite the maximization problem as

$$V(Y_t) = \max\left\{u(Y_t), \lambda\, dt\, [u(Y_t) - V(Y_t)] + E[V(Y_t + dY_t)|Y_t]\right\}, \tag{7}$$

where $u(Y_t)$ is the value function from liquidating immediately, and $\lambda\, dt\, [u(Y_t) - V(Y_t)] + E[V(Y_t + dY_t)|Y_t]$ is the value function from continuing the project. Note that $\lambda\, dt$ is the probability that the project naturally pays off over the period $(t, t+dt)$, and $u(Y_t) - V(Y_t)$ is the change to the value function in such an event. [10] Since the arrival process of the natural payoff is exogenous and independent of Y, the decision of whether to liquidate or continue the project can be based on the expectation of the arrival of the natural payoff process.

In the region where $u(Y) > \lambda\, dt\, [u(Y_t) - V(Y_t)] + E[V(Y_t + dY_t)|Y_t]$, the agent would liquidate the project immediately, and we call this region the stopping region. In the complement of the stopping region, the agent would prefer to continue the project, and we call this region the continuation region. In the continuation region, we have

$$V(Y_t) = \lambda\, dt\, [u(Y_t) - V(Y_t)] + E[V(Y_t + dY_t)|Y_t]. \tag{8}$$

[8] It will become clear later that our results are driven by the kink and the convexity of the agent's utility function. Because the agent in our model is risk seeking when the project is in losses, he delays the liquidation of the project. Also because the agent is most risk averse at the kink, he liquidates the project at the break-even point. We believe that these basic results would still go through even if more general utility functions are adopted as long as these utility functions are convex in the loss region and exhibit a kink at the origin.

[9] Introducing a positive discount rate, say r, would only cause a minor change to our analysis. More specifically, Eq. (9) would become

$$\tfrac{1}{2}\sigma^2 V_{YY} + (\mu - c)V_Y - (r + \lambda)V + \lambda u(Y) = 0.$$

Solving this equation presents no additional challenge than the original equation.

[10] If the manager continues the project, his expected utility is given by

$$\begin{aligned}\lambda u(Y_t)\, dt + (1 - \lambda\, dt)E[V(Y_t + dY_t)|Y_t] &= \lambda u(Y_t)\, dt - \lambda\, dt\, E[V(Y_t + dY_t)|Y_t] \\ &\quad + E[V(Y_t + dY_t)|Y_t]\, dt \\ &= \lambda\, dt[u(Y_t) - V(Y_t)] + E[V(Y_t + dY_t)|Y_t].\end{aligned}$$

If $V(Y)$ is smooth, the use of Ito's lemma gives the following differential equation in the continuation region:

$$\tfrac{1}{2}\sigma^2 V_{YY} + (\mu - c)V_Y - \lambda V + \lambda u(Y) = 0. \tag{9}$$

In the stopping region, we have that $V(Y) = u(Y)$.

The following two parameters characterize the fundamental of the project:

$$v = \frac{\mu - c}{\sigma^2}, \quad \rho = \frac{\lambda}{\sigma^2}. \tag{10}$$

Intuitively, v represents the risk adjusted growth rate ("Sharpe ratio") of the project, while ρ is a "variance-adjusted arrival rate" of natural liquidation obtained by re-scaling time so that $\sigma^2 = 1$. We can rewrite the differential equation using v and ρ:

$$\tfrac{1}{2} V_{YY} + vV_Y - \rho V + \rho u(Y) = 0. \tag{11}$$

This equation is non-homogenous, and its homogenous part is $1/2V_{YY} + vV_Y - \rho V = 0$. The general solution of the homogenous part is given by

$$V = A_1 e^{a_1 Y} + A_2 e^{a_2 Y}, \tag{12}$$

where A_1 and A_2 are two constants, and a_1 and a_2 represent the two roots of the quadratic equation $1/2a^2 + va - \rho = 0$, which are given by

$$a_1 = -v + \sqrt{v^2 + 2\rho} > 0, \quad a_2 = -v - \sqrt{v^2 + 2\rho} < 0. \tag{13}$$

The general solution of the non-homogeneous differential equation (11) is the general solution of its homogeneous part plus any of its specific solutions. In the following sections, we will solve this equation under different boundary conditions. The challenge lies in the determination of these boundary conditions.

3. Benchmark case: the standard exponential utility

As a benchmark, we first discuss the liquidation decision of an agent with a standard one-piece exponential utility function:

$$u_0(Y) = \phi_1 \left(1 - e^{-\gamma_1 Y}\right), \quad \forall\, Y \in (-\infty, \infty). \tag{14}$$

If the agent holds the project until it pays off naturally, his value function is a solution of Eq. (11), with $u(Y)$ replaced by $u_0(Y)$ in Eq. (14). The solution is given by

$$V_0(Y) = \phi_1 \left(1 - g_1 e^{-\gamma_1 Y}\right), \quad \text{with } g_1 = \frac{\rho}{\rho + v\gamma_1 - \frac{1}{2}\gamma_1^2}. \tag{15}$$

In order to prevent $V_0(Y)$ from exploding (or equivalently $g_1 > 0$), the following restrictions on the project's Sharpe ratio is needed for the benchmark case:

$$v > \frac{1}{2}\gamma_1 - \frac{\rho}{\gamma_1}. \tag{16}$$

This restriction must be imposed because the standard (one-piece) exponential utility function is un-bounded from below. However, it is not necessary for the bounded two-piece exponential function of prospect theory.

If the agent exercises his liquidation option, he realizes a value function of $V(Y) = u_0(Y)$. It is easy to see the following: If $g_1 < 1$ or, equivalently, $v > \frac{1}{2}\gamma_1$, then $u_0(Y) < V_0(Y),\ \forall Y \in (-\infty, \infty)$, which indicates that it is never optimal to liquidate voluntarily. On the other hand, if $g_1 \geqslant 1$ or, equivalently, $v \leqslant \frac{1}{2}\gamma_1$, liquidating immediately is preferable when compared with never liquidating voluntarily. However, the optimal strategy could be to liquidate at some point in the future instead of liquidating immediately. It is important to notice that the process $u_0(Y)$ is a supermartingale if the Sharpe ratio $v \leqslant \frac{1}{2}\gamma_1$. This is because

$$\begin{aligned} E_t[du_0(Y)] &= \left[\tfrac{1}{2}\, u_{0,YY} + v\, u_{0,Y}\right] \sigma^2\, dt \\ &= -\gamma_1 \left(\tfrac{1}{2}\gamma_1 - v\right) e^{-\gamma_1 Y} \sigma^2\, dt \leqslant 0, \quad \forall\, v \leqslant \tfrac{1}{2}\gamma_1. \end{aligned} \tag{17}$$

Suppose that the optimal strategy is to liquidate voluntarily according to a stopping time τ_2. The resulting expected value function is then given by

$$V(Y_t) = E_t[u_0(Y_{t+\min\{\tau_1,\tau_2\}})] \leqslant u_0(Y_t), \tag{18}$$

where the inequality arises from the supermartingale property of the $u_0(Y)$ process. Therefore, the optimal strategy is to liquidate immediately if $v \leqslant \frac{1}{2}\gamma_1$. The next proposition summarizes the optimal strategy for an agent with a standard exponential utility.

Proposition 1. *For an agent with a standard exponential utility function specified in Eq.* (14), *the optimal strategy is to liquidate the project immediately if its Sharpe ratio* $v \leqslant \frac{1}{2}\gamma_1$, *or never to liquidate if its Sharpe ratio* $v > \frac{1}{2}\gamma_1$.

In the next section, we discuss the liquidation decision under prospect theory. Since prospect theory suggests that the agent's risk aversion changes with the paper gain and loss of the project, the changes in risk aversion could induce very different liquidation decisions, although the Sharpe ratio of the project dose not change over time.

4. Liquidation decisions under prospect theory

In this section, we analyze the liquidation decisions under prospect theory. With the two-piece exponential utility function specified in Eq. (5), the optimal liquidation strategy depends on four cases that arise from different values of the Sharpe ratio of the project, v.

To facilitate our discussion, we first show the following propositions regarding the local properties of the expected growth rate of the agent's direct utility $u(Y)$:

Proposition 2. (a) *If* $v > \frac{1}{2}\gamma_1$, *the process* $u(Y)$ *with* $Y > 0$ *is a submartingale*; *otherwise*, *it is a supermartingale*. (b) *If* $v > -\frac{1}{2}\gamma_2$, *the process* $u(Y)$ *with* $Y < 0$ *is a submartingale*; *otherwise*, *it is a supermartingale*.

Proposition 2 has direct implications for the agent's liquidation decisions. Let us first consider a project that is currently in gains ($Y > 0$). If its Sharpe ratio is higher than $\frac{1}{2}\gamma_1$, the agent would not want to liquidate the project immediately, since $u(Y)$ is a submartingale. On the other hand, if the project's Sharpe ratio is lower than $\frac{1}{2}\gamma_1$, the agent would prefer to liquidate immediately unless the optimal strategy is to wait after Y becomes non-positive, since $u(Y)$ is a supermartingale in this case. Next, we consider a project that is currently in losses ($Y < 0$). If the project's Sharpe ratio is higher than $-\frac{1}{2}\gamma_2$, the agent would not want to liquidate it immediately. On the other hand, if the project's Sharpe ratio is lower than $-\frac{1}{2}\gamma_2$, the agent would prefer to liquidate immediately unless the optimal strategy is to wait after Y hits zero.

To fully understand the exact optimal strategy, we need to compare the expected value function from never liquidating voluntarily with the utility from liquidating immediately, especially at the reference point. We define the expected value function from never liquidating voluntarily as

$$V_0(Y_t) = E_t[u(Y_{t+\tau_1})], \tag{19}$$

where τ_1 represents the Poisson arrival time of natural liquidation. The derivation of $V_0(Y)$ through direct expectations can be tedious. However, V_0 is a solution to ordinary differential equation (11) with the boundary conditions that $V_0(-\infty) = -\phi_2$ and $V_0(\infty) = \phi_1$. The solution is summarized in the following proposition.

Proposition 3. *$V_0(Y)$ is monotonically increasing with both Y and v, and it is given by*

$$V_0(Y) = \begin{cases} \frac{\phi_2(g_2-1)a_1-\phi_2 g_2\gamma_2-\phi_1(1-g_1)a_1+\phi_1 g_1\gamma_1}{a_1-a_2} e^{a_2 Y} + \phi_1\left(1 - g_1 e^{-\gamma_1 Y}\right) & \text{if } Y \geqslant 0 \\ \frac{\phi_1 g_1\gamma_1-\phi_1(1-g_1)a_2-\phi_2 g_2\gamma_2+\phi_2(g_2-1)a_2}{a_1-a_2} e^{a_1 Y} + \phi_2\left(g_2 e^{\gamma_2 Y} - 1\right) & \text{if } Y < 0, \end{cases} \tag{20}$$

where g_1 is given in Eq. (15) *and g_2 is given by*

$$g_2 = \frac{\rho}{\rho - v\gamma_2 - \frac{1}{2}\gamma_2^2}. \tag{21}$$

For tractability of proofs, we impose that

$$\text{if } v = \gamma_1/2, \quad V_0(0) < u(0) = 0, \tag{22}$$

which implies that for a marginal project in the benchmark case, the agent's expected value function at the reference point in the case where he never liquidates voluntarily, is smaller than that in the case where he liquidates immediately. This is equivalent to

$$\phi_2\gamma_2 - \phi_1\gamma_1 > \frac{\phi_2\gamma_2(\gamma_1+\gamma_2)}{2\rho - \gamma_2(\gamma_1+\gamma_2)}\left(\sqrt{\gamma_1^2/4 + 2\rho} - \gamma_1/2 - \gamma_2\right) \tag{23}$$

whose right-hand side is always positive.[11] This restriction requires that the kink (the slope differential) in the agent's direct value function across the reference point is larger than certain value, given on the right-hand side of inequality (23). Economically, when $v = \frac{\gamma_1}{2}$, $g_1 = 1$. Consequently, under the standard exponential utility function, the manager would be indifferent between holding and liquidating the project. Under the prospect theory value function, the manager is more sensitive to losses than gains at the reference point, he would liquidate the project so as to avoid losses that hurt him proportionally more than the same amount of gains. There are four cases for the optimal liquidation strategy.

Case I: $V_0(0) > u(0)$ *and* $v > \frac{1}{2}\gamma_1$.

In this case, the Sharpe ratio of the project is very high. The optimal strategy in this case is never to liquidate voluntarily. According to our discussion before, when the project is either in gains ($Y > 0$) or in losses ($Y < 0$), the agent would not want to liquidate immediately. While on the reference point, the agent prefers never liquidating voluntarily to liquidating immediately $[V_0(0) > u(0)]$, therefore she would never liquidate voluntarily.

Case II: $V_0(0) < u(0)$ *and* $v > \frac{1}{2}\gamma_1$.

In this case, the Sharpe ratio of the project is sufficiently high so that an agent with a standard exponential utility function would never liquidate the project. But the Sharpe ratio is not high enough so that an agent with a prospect value function would liquidate the project at the reference point.[12] According to our discussion earlier, the agent would not liquidate the project when it is either in gains ($Y > 0$) or in losses ($Y < 0$). On the other hand, at the reference point the agent prefers liquidating the project immediately to never liquidating it voluntarily. Intuitively, the optimal strategy is to liquidate only at the reference point as summarized in the following proposition.

Proposition 4. *If* $V_0(0) < u(0)$ *and* $v > \frac{1}{2}\gamma_1$, *the optimal strategy is to liquidate the project voluntarily only when it breaks even, and the resulting expected value function is*

$$V(Y) = \begin{cases} -\phi_1(1-g_1)e^{a_2 Y} + \phi_1(1 - g_1 e^{-\gamma_1 Y}) & \text{if } Y \geqslant 0 \\ \phi_2(1-g_2)e^{a_1 Y} + \phi_2(g_2 e^{\gamma_2 Y} - 1) & \text{if } Y < 0. \end{cases} \tag{24}$$

Case III: $V_0(0) < u(0)$ *and* $-\frac{1}{2}\gamma_2 < v < \frac{1}{2}\gamma_1$.

In this case, the Sharpe ratio of the project is not high enough for an agent with a standard exponential utility function to hold the project, but it is not low enough for a prospect agent to liquidate the project immediately. According to our discussion following Proposition 2, in this case, $u(Y)$ is a submartingale when the project is in losses ($Y < 0$), while it is

[11] If $2\rho - \gamma_2(\gamma_1 + \gamma_2) > 0$ or $2\rho > \gamma_2(\gamma_1 + \gamma_2)$, then we have

$$\sqrt{\frac{\gamma_1^2}{4} + 2\rho} - \frac{\gamma_1}{2} - \gamma_2 > \sqrt{\frac{\gamma_1^2}{4} + \gamma_2(\gamma_1 + \gamma_2)} - \frac{\gamma_1}{2} - \gamma_2 = \sqrt{\left(\frac{\gamma_1}{2} + \gamma_2\right)^2} - \left(\frac{\gamma_1}{2} + \gamma_2\right) = 0.$$

Similarly, we can show that if $2\rho < \gamma_2(\gamma_1 + \gamma_2)$, then

$$\sqrt{\frac{\gamma_1^2}{4} + 2\rho} - \left(\frac{\gamma_1}{2} + \gamma_2\right) < 0.$$

[12] Recall that V_0 is an increasing function of both Y and v, $V_0(0) < u(0)$ imposes an upper bound on v.

a supermartingale when the project is in gains ($Y > 0$). Therefore, if the project is in gains, the agent would choose either to liquidate it immediately or to wait to liquidate it unless Y becomes non-positive; while if the project is in losses, the agent would not want to liquidate it immediately. Since, at the reference point, liquidating immediately is preferred to never liquidating, the optimal strategy is to wait until Y hits zero to liquidate if the project is currently in losses, or to liquidate immediately if the project is currently in gains, as summarized in the following proposition.

Proposition 5. *If $V_0(0) < u(0)$ and $-\gamma_2/2 < v \leqslant \gamma_1/2$, then the optimal liquidation strategy is to wait until Y hits zero for the first time if the project is currently in losses ($Y < 0$), or to liquidate immediately if the project is currently in gains ($Y \geqslant 0$). The agent's value function is given by*

$$V(Y) = \begin{cases} \phi_1(1 - e^{-\gamma_1 Y}) & \text{if } Y \geqslant 0, \\ \phi_2(1 - g_2)e^{a_1 Y} + \phi_2(g_2 e^{\gamma_2 Y} - 1) & \text{if } Y < 0. \end{cases} \tag{25}$$

Case IV: $V_0(0) < u(0)$ and $v < -\frac{1}{2}\gamma_2$.

In this case, since $u(Y)$ is always a supermartingale according to Proposition 2, and $V_0(0) < u(0)$, the optimal strategy is to liquidate immediately.

5. Further discussion

Our discussion of the disposition effect requires a key assumption of narrow framing, as discussed and demonstrated in Barberis and Huang [2] and particularly in Barberis, Huang, and Thaler [4]. Because firms or investors typically have more than one project or asset in their portfolios, the theoretical results based on one asset, as obtained in our model as well as in almost all other models, may be used to interpret empirical and experimental findings only when narrow framing is assumed. Fortunately, Barberis, Huang, and Thaler have shown that narrow framing may be an important feature of decision-making under risk. Nevertheless, it could be interesting to reconsider the problem when there are multiple investments that can be liquidated at will and when these investments are partially or fully integrated.

Prospect theory as advanced by Kahneman and Tversky [16] is a theory of choice under uncertainty in one-shot games. In the spirit of a one-shot game in which consumption takes place only once (e.g., when the agent liquidates the project), our model discusses the disposition effect in a formal manner. We find that under certain conditions, an agent with a prospect theory utility function delays (compared to a risk-averse agent) the liquidation when the project is in losses. A key point of this one-shot game is that paper (unrealized) gains and losses do not affect the agent's utility, or, in other words, only when the agent realizes the gains and losses is his utility affected. This assumption is different from Barberis et al. [3], who allow unrealized paper gains and losses to affect an agent's utility. More specifically, they make two assumptions. First, the investor is more sensitive to paper losses than to paper gains due to loss aversion. Second, after a prior loss, the investor becomes more risk averse: after being burnt by the initial loss, he is more sensitive to additional setbacks. Note that the second assumption, which is motivated by an earlier study by Thaler and Johnson [26], could generate an incentive for the agent to liquidate early in the presence of earlier losses, because the agent is more risk averse in these situations. It would be interesting to extend the current model to incorporate the additional effects caused by unrealized gains and losses, along the lines of Barberis et al. [3].

A restriction on the original Kahneman–Tversky [16] experiment is that quantities are not choice variables. As a result, the issue of partial liquidation is not addressed by the extant literature. The possibility of partial liquidation is discussed in neither Shefrin and Statman [24] nor Odean [21]. Given this background, our results capture the spirit of the disposition effects illustrated in those studies for financial assets and may be directly used to discuss the empirical studies by Genesove and Mayer [11] and Shefrin [23] in liquidating housing positions and real projects. Genesove and Mayer find evidence that house sellers, who have incurred losses, are reluctant to realize them. Shefrin presents the case of Sony Corporation in which after a project had suffered heavy losses, the co-founder in charge continued to invest in it and refused to accept a sure loss. In considering houses and firm projects, the exclusion of partial liquidation is perhaps an acceptable assumption, but it would be of interest to incorporate partial liquidation in our model.

In Kahneman and Tversky [16] and Tversky and Kahneman [27], agents transform probabilities nonlinearly, overweighting small probabilities and underweighting moderate and high probabilities. It would be intractable for us to obtain solutions under nonlinear probability transformation. For simplicity, we have omitted this feature of prospect theory. We believe that our results will probably not be affected much because they are mostly driven by the convexity in the agent's prospect value function.

6. Conclusion

This paper provides a formal framework to analyze the liquidation decisions of economic agents under prospect theory. Our model indicates two forces in play. On the one hand, the convexity in the agent's direct value function of losses can induce the agent to delay liquidation. On the other hand, the loss aversion (the agent's higher sensitivity to losses than to gains) induces the agent to be more risk averse near the reference point, and therefore can induce liquidation near this point. Consistent with the disposition effect documented in the empirical and experimental studies, we find that prospect theory induces agents to delay liquidation of relatively inferior projects if they are in paper losses, and, on the other hand, it induces agents to accelerate liquidation of relatively superior projects due to his loss aversion. In particularly, we find a tendency to liquidate exactly at the break-even point.

Technically, this paper solves an optimal stopping problem with a non-smooth objective function at the origin, using the dynamic programming approach. It is illustrated that the non-differentiability of the value function at the origin plays a key part in the agent's decision rules and that the usual smooth pasting conditions, developed in the previous studies for smooth objective functions, do not apply. Instead, we find liquidating at the origin as a corner solution, which is shown to be optimal through the verification theorems.

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Appendix A. Proofs

A.1. Proof of Proposition 2

(a) For $Y > 0$,

$$E_t[du(Y)] = \left[\tfrac{1}{2}\, u_{YY} + v\, u_Y\right] \sigma^2\, dt = -\gamma_1 \left(\tfrac{1}{2}\gamma_1 - v\right) e^{-\gamma_1 Y} \sigma^2\, dt. \tag{A.1}$$

Thus, if $v > \frac{1}{2}\gamma_1$, $E_t[du(Y)] > 0$. This implies that $u(Y)$ is expected to grow, and therefore $u(Y)$ is a submartingale for $Y > 0$. On the other hand, if $v < \frac{1}{2}\gamma_1$, $E_t[du(Y)] < 0$, and therefore $u(Y)$ is a supermartingale for $Y > 0$. [13]

(b) For $Y < 0$,

$$E_t[du(Y)] = \left[\tfrac{1}{2}\, u_{YY} + v\, u_Y\right] \sigma^2 dt = \gamma_2 \left(\tfrac{1}{2}\gamma_2 + v\right) e^{\gamma_2 Y} \sigma^2\, dt. \tag{A.2}$$

Thus, if $v > -\frac{1}{2}\gamma_2$, $E_t[du(Y)] > 0$. This implies that $u(Y)$ is expected to grow, and therefore $u(Y)$ is a submartingale for $Y < 0$. On the other hand, if $v < -\frac{1}{2}\gamma_2$, $E_t[du(Y)] < 0$, and therefore $u(Y)$ is a supermartingale for $Y < 0$.

A.2. Proof of Proposition 3

$Y_{t+\tau_1}$ in the expectation of Eq. (19) has a Gaussian distribution with a mean of $Y_t + v\sigma^2\tau_1$ and a variance of $\sigma^2\tau_1$. Since function $u(\cdot)$ is strictly increasing, $E_t[u(Y_{t+\tau_1})]$ increases with the mean of $Y_t + v\sigma^2\tau_1$, and therefore $V_0(Y)$ increases with Y and v. To derive V_0, we solve Eq. (11) for both $Y > 0$ and $Y < 0$. If $Y > 0$, the general solution is

$$V_0 = A_1 e^{a_1 Y} + A_2 e^{a_2 Y} + \phi_1 \left(1 - g_1 e^{-\gamma_1 Y}\right), \tag{A.3}$$

where $\phi_1 \left(1 - g_1 e^{-\gamma_1 Y}\right)$ can be verified as a specific solution, and A_1 and A_2 are constants to be determined. If $Y < 0$, the general solution is

$$V_0 = B_1 e^{a_1 Y} + B_2 e^{a_2 Y} + \phi_2 \left(g_2 e^{\gamma_2 Y} - 1\right), \tag{A.4}$$

[13] In the experimental studies by Weber and Camerer [28], subjects make portfolio decisions among 6 risky assets before each of the 14 periods. They find that subjects tend to sell winners and keep losers. When the assets are automatically sold after each period, or the gains and losses of each period are forced to be realized at the end of the period, the disposition effect is greatly reduced. See also [18].

where $\phi_2\left(g_2 e^{\gamma_2 Y}-1\right)$ can be shown to be a specific solution, and B_1 and B_2 are constants to be determined. We have the following boundary conditions for V_0:

$$V_0(-\infty)=-\phi_2,\quad V_0(\infty)=\phi_1,\quad V_0(0-)=V_0(0+),\quad V_0'(0-)=V_0'(0+), \tag{A.5}$$

which can be used to determine constants A_1, A_2, B_1 and B_2.

A.3. Proof of Proposition 4

Let $\tau_2=\min\{T|Y_T=0\}$. The expected value function from the strategy, which is to voluntarily liquidate the project once Y hits zero, is then given by

$$V(Y)=E[u(Y_{t+\min\{\tau_1,\tau_2\}})]. \tag{A.6}$$

Since $u(Y)$ is a submartingale when $v>\frac{1}{2}\gamma_1$, we have

$$V(Y)\geqslant V(0)=u(0),\quad \forall Y. \tag{A.7}$$

To specifically derive $V(Y)$, we know that $V(Y)$ must satisfy the differential equation (11) with the boundary conditions that

$$V(-\infty)=-\phi_2,\quad V(0)=0,\quad V(\infty)=\phi_1. \tag{A.8}$$

For $Y\geqslant 0$, the general solution is given in Eq. (A.3); while for $Y\leqslant 0$, the general solution is given in Eq. (A.4). By fitting the boundary conditions, we can derive the constants A_1, A_2, B_1 and B_2, and the expected value function is given in Eq. (24).

Note that $V(Y)$ is not differentiable at $Y=0$, similar to the direct utility function. Around this point,

$$V'(0+)=-\phi_1(1-g_1)a_2+\phi_1 g_1\gamma_1,\quad V'(0-)=\phi_2(1-g_2)a_1+\phi_2 g_2\gamma_2. \tag{A.9}$$

The difference between them is given by $V'(0+)-V'(0-)=(a_1-a_2)V_0(0)<0$, due to $V_0(0)<0$ by assumption. Therefore, there is a concave kink in $V(Y)$ at $Y=0$. This kink is the reason that the agent liquidates at $Y=0$, which is to avoid the concave kink.

Next, we show that the stopping time $\tau_2=\min\{T|Y_T=0\}$ is the optimal liquidation strategy. Consider the expected value function from any stopping time τ_2'. According to the Tanaka formula (the generalized Ito rule for non-differentiable functions):

$$\begin{aligned} E_t[V(Y_{t+\tau_2'})]=V(Y_t)+E_t\Bigg\{&\int_t^{t+\tau_2'}[\sigma^2 V_{YY}/2+(\mu-c)V_Y+\lambda(u-V)]\,ds\\ &+L_{\tau_2'}(0)[V'(0+)-V'(0-)]\Bigg\}, \end{aligned} \tag{A.10}$$

where $L_{\tau_2'}(0)$ is the local time that the Y_t process spends at the reference point. The non-differentiability of $V(Y)$ at $Y=0$ introduces the term of $L_{\tau_2'}(0)[V'(0+)-V'(0-)]$.[14] Since $V(Y)$ satisfies differential equation (11) almost everywhere, the integral term in

[14] See, e.g., Section 3.6 of Karatzas and Shreve [17] for more details on the Tanaka formula and local time.

Eq. (A.10) is always zero. Since the local time $L_{\tau_2'}(0)$ is a non-negative and non-decreasing process and since $V'(0+) - V'(0-) < 0$, $E_t[V_{t+\tau_2'}] \leqslant V(Y_t)$ for any stopping time τ_2'.[15] In addition, since $V(Y) \geqslant u(Y)$ for any Y, we have $V(Y) \geqslant E[u(Y_{t+\tau})]$, with $\tau = \min(\tau_1, \tau_2')$ for any stopping rule τ_2'. As a result, there does not exist any other stopping rule that can deliver a better expected value function than the one from always liquidating at zero.

A.4. Proof of Proposition 5

The value function $V(Y)$ satisfies the differential equation (11) for $Y < 0$, and $V(Y) = u(Y)$ for $Y > 0$. In addition, it has the following boundary conditions: $V(-\infty) = -\phi_2$ and $V(0) = u(0) = 0$. Solving the differential equation and the boundary conditions in the same way as before gives the expected value function given in Eq. (25).

The value function in Eq. (25) is smooth everywhere except that it is not differentiable at $Y = 0$. It is easy to see that

$$V'(0-) = \phi_2(1 - g_2)a_1 + \phi_2 g_2 \gamma_2, \quad V'(0+) = \phi_1\gamma_1, \tag{A.11}$$

and the difference of the derivatives around zero is

$$V'(0-) - V'(0+) = (\phi_2\gamma_2 - \phi_1\gamma_1) - \phi_2(g_1 - 1)(a_1 - \gamma_2). \tag{A.12}$$

If we treat a_1 and g_2 as functions of v, direct differentiation can show that $(g_1 - 1)(a_1 - \gamma_2)$ increases with v in $(-\gamma_2/2, \gamma_1/2)$. Note that when $v = \gamma_1/2$, $\phi_2(g_1 - 1)(a_1 - \gamma_2)$ is exactly the right-hand side of inequality (23), therefore

$$V'(0-) - V'(0+) > 0, \quad \forall\, v \in (-\gamma_2/2, \gamma_1/2) \tag{A.13}$$

which indicates a concave kink at $Y = 0$ in $V(Y)$. Also note that the threshold strategy as specified in Proposition 3 yields

$$V(Y) > u(Y), \ \ \forall Y < 0; \quad V(Y) = u(Y), \ \ \forall Y \geqslant 0. \tag{A.14}$$

In addition,

$$\sigma^2 V_{YY}/2 + (\mu - c)V_Y + \lambda(u - V) \begin{cases} = 0 & \text{if } Y < 0 \\ < 0 & \text{if } Y \geqslant 0. \end{cases} \tag{A.15}$$

Consider the expected value of V at any future stopping time τ_2' implied by Tanaka formula in Eq. (A.10). Since the integral term and the term involving the local time are both non-positive, we obtain $E_t[V_{t+\tau_2'}] \leqslant V(Y_t)$. On the other hand, $V(Y) \geqslant u(Y)$ for any Y, consequently, $V(Y_t) \geqslant E[u_\tau]$ with $\tau = \min(\tau_1, \tau_2')$. Therefore, there does not exist any other stopping rule that can deliver a better value function than $V(Y)$.

[15] Intuitively, the concave kink at zero always decreases the expectation as implied by Jensen's inequality, which is precisely given by the term with local time.

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