Locked Wealth, Subjective Valuation and Managerial Hedging under High-Water Marks: A Structural Model

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First Draft: January, 2016

This Draft: July 20, 2016

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Abstract We consider a hedge fund manager who operates the hedge fund asset and her private portfolio simultaneously under high-water mark compensation in an incomplete market. In her private portfolio choice problem, she is assumed to lock a constant portion of her private wealth in the hedge fund as managerial ownership. In our model, the claim in the hedge fund provides implicit incentive of valuation maximization, since the manager acts in the interest of the management team. Thus, manager takes the leverage choice in the hedge fund as given and derives optimal private portfolio choice as well as the subjective martingale pricing kernel which is used to evaluate her claim in the hedge fund wherein she derives the optimal leverage to maximize her claim valuation. By virtue of subjective valuation, we ingeniously evade introducing managerial ownership as an extra state variable since in our model managerial ownership only affects the location of subjective valuation interval on the normalized state space. A closed form equation for the manager's subjective valuation function, as well as optimal leverage choice, is derived. In the full equilibrium dynamics, we find that managerial hedging incentive induces the investors to require a considerable portion of the manager's locked wealth in order to initiate the fund. The optimal leverage in the fund decreases with the manager's locked wealth ratio. Importantly, market beta and idiosyncratic volatility play different roles in affecting optimal leverage due to managerial hedging behavior.

JEL Classification G11, G12, G2, G32

1 Introduction

High-water mark (henceforth HWM) provision is a special compensation structure that is commonly used in the hedge fund industry. Given a HWM compensation, a fund manager's incentive fee is collected conditional on she makes up all the past losses in fund total asset under management (henceforth AUM) so that the asset surpasses the historically highest AUM level. An intuitive interpretation of HWM compensation is to view the incentive fee as a series of call options whose strike prices are set at HWM levels. Besides HWM compensation, managerial ownership is also a common practice in hedge funds.[1] Theoretically, when discussing managerial ownership, the finance literature generally ignores the effects of the manager's private wealth on managerial ownership. The fund management problem is considered in a complete market where both the amount of managerial ownership and the flows of compensation can be perfectly replicated by the manager when she invests private wealth. Thus, the effect of managerial hedging in a complete market is tenuous due to a separation between the manager's private wealth management and consumption decision from her fund portfolio choice problem. Nevertheless, even in complete market, the role of managerial ownership depends on its relative magnitude to the manager's private wealth. Intuitively, when managerial ownership only accounts for a minute portion of the manager's private wealth, the incentive alignment effect of managerial ownership will hardly become relevant: A richer manager may bear heavier opportunity cost or enjoy stronger bargaining power and thus her individual rationality condition may be too prohibitive to hold. Furthermore, when modeling hedge funds, the assumption of complete market is not apposite: Shares of the hedge fund are not traded in the market and positions of the fund asset are confidential so that outside investors can not simply replicate its returns.

This paper is devoted to the question: What is the interaction between the manager's private portfolio choice problem and the hedge fund operation in an incomplete market when the fund manager is contractually locked in the fund and faces implicit incentives in the fund? In the presence of managerial ownership, a series of other interesting research questions arise based on this question: How does managerial hedging behavior bias the manager's private portfolio choice problem from the standard Merton's problem? What is the effect of locked wealth on the fund's operation such as investment decisions? Does the manager display different risk appetite for systematic risk and idiosyncratic risk?

We consider a hedge fund manager who operates the hedge fund asset and her private portfolio

[1] According to Agarwal, Daniel, and Naik (2009), the average rate of managerial ownership in the hedge fund industry is about 7.1%. Although there is a paucity of managerial ownership data, they estimate the managerial ownership of each fund by calculating dollar amount of incentive fee that is reinvested into the hedge fund and dividing it by the fund's total asset. This method, however, may underestimate managerial ownership.

simultaneously. At the commencement of the model economy, a group of investors reach an investment contract with a fund manager to establish a hedge fund. Both the investors and manager contribute part of their wealth to form the AUM of the fund. Throughout the duration of the fund, the manager is contracted to lock a constant portion of her private wealth in the hedge fund as managerial ownership. Then the manager, together with the management team, is compensated by a HWM provision. We do not model management fee, since in practice, management fees are often reimbursed for administration expenses. Instead, these expenses together with any other dividends or withdrawal are modeled as a constant rate of AUM outflows. The locked wealth clause is implemented by share transaction between the manager and the investors which is settled by the AUM accounting numeraire. Thus, it imposes a natural boundary of the investment contract: When the fund AUM drops or manager's wealth surges to a degree that managerial ownership is equal to 100%, the manager totally privatizes the fund asset and the investment contract ceases. Moreover, we assume that the manager is able to generate alpha returns when managing the AUM by constructing a special risky asset (i.e. a trading strategy) which incurs both market risk and idiosyncratic risk. Since the risky asset which is special and confidential cannot be replicated by outside agents in the economy; in this sense, our model economy is incomplete. On the other hand, when the manager makes her private portfolio choice decisions, she invests her unlocked wealth between the market portfolio and the risk-free bond. Under continuous time setting, since the HWM compensation is collected only on a negligible set over the time horizon, the manager's private portfolio choice problem is actually a constrained Merton's problem almost everywhere, which, given leverage choice in the hedge fund, is solved in closed form with an important by-product of the manager's subjective martingale pricing kernel. Note that the optimal leverage choice in the hedge fund is not solved in the manager's private portfolio choice problem since her claim in the hedge fund bestows implicit incentive of valuation maximization for the manager to act in interest of the management team which is not explicitly modeled in the private portfolio choice problem since the timing of HWM compensation is of zero measure on the time dimension under continuous time setting. Instead, we follow the equilibrium valuation approach in Goetzmann, Ingersoll, and Ross (2003, GIR henceforth) and among others, to evaluate the manager's claim in the hedge fund: Discounted by subjective martingale pricing kernel, the manager's subjective valuation of her claim should be a martingale under optimal leverage control. A closed form subjective valuation function is solved. Since the investors are able to diversify the idiosyncratic volatility in their investment by only purchasing a limited amount of shares in the fund and thus holding diversified portfolios, they evaluate their claims by discounting it at the market martingale pricing kernel. We further derive participation constraint conditions that both the manager and investors require to enter into the investment contract. Under this model setting, we develop full equilibrium dynamics in our model economy.

By virtue of subjective valuation, we ingeniously evade introducing managerial ownership as an extra state variable in the hedge fund valuation since in our model managerial ownership only affects the location of subjective valuation interval on the normalized state space. In the full equilibrium dynamics, the manager is trying to hedge away market exposure of her locked wealth in the hedge fund AUM, which results in essentially Merton's portfolio weights between market portfolio and risk-free bond. However, due to lock wealth clause, the idiosyncratic risk and alpha return still influence the private wealth process. The manager's private consumption is biased from that in a standard Merton's problem although the consumption rate is still a constant.

The manager's subjective martingale pricing kernel differs from its market counterpart in two ways: First, the interest rate in the subjective martingale pricing kernel becomes the subjective interest rate that is adjusted by certainty equivalent return on the manager's locked wealth in the fund asset. Second, idiosyncratic risk is priced in the subjective martingale pricing kernel. With this subjective martingale pricing kernel, the manager derives a power-type subjective valuation function of her claim in the fund, whose concavity depends on parameter values. As an example, in our model calibration, we show that the manager's subjective valuation is convex while the investors' market valuation is concave. However, due to the complex form of leverage in our model, the concavity is no longer a suitable measure for the manager's risk preference. And thus, one should be cautious to draw conclusions on how the investment contract affects the manager's implicit risk preference.

Importantly, even compensated by HWM provision with a convex structure, the manager in our model choose a constant leverage in the hedge fund as that in Panageas and Westerfield (2009, henceforth PW), since investment contract in our model does not specify a definite investment maturity. In our calibrated example, the optimal leverage decreases with the proportion of locked wealth, showing incentive alignment effect. Interestingly, the investors require a substantial minimum locked wealth of 48.28% to participate in the investment contract. We further show that idiosyncratic risk and market risk play different roles in impacting optimal leverage. Since idiosyncratic risk cannot be diversified by the manager in her private wealth, her risk aversion motivation induces a deleveraging reaction to an increase in idiosyncratic volatility. On the other hand, managerial hedging makes the manager attached to market loading in the hedge fund AUM. Since the market exposure can be totally hedged away in her private wealth, the manager in effect leverages up in response to a higher market beta in the fund so that she can better off by enjoying the alpha return. Thus, our model provides an important policy advice for the hedge fund industry: To prevent the manager's risk shifting incentive under a general convex compensation circumstance, managerial ownership is efficacious only when the manager cannot hedge the risk sourced from that ownership.

Our work is linked with several strands of the literature. First, as a hedge fund model, we share continuous time hedge fund modeling framework developed first in GIR in which the valuation approach is introduced. PW further prove that the valuation approach is equivalent to a present value maximization problem. As a matter of fact, in case of no managerial ownership or locked wealth clause, the subjective martingale pricing kernel reduces to the market one and the hedge fund valuation in our structural model coincides with the model in PW.[2] Following PW, Drechsler (2014) is another seminar work that enriches the model in PW by adding liquidation boundary and manager's outside option. Lan, Wang, and Yang (2013, henceforth LWY) provide a general model that incorporate many hedge fund characteristics. These works generally assume away concavity in the manager's utility and calculate market valuations of claims in the fund. Due to the zero measure of time set when the manager collects incentive fee under continuous time setting, it is technically difficult to introduce risk aversion parameter when modeling hedge funds. By assuming that income flows from HWM can be accumulated at risk-free rate and consumed at a terminal date by the manager[3], Guasoni and Obłój (2013) solve a certainty equivalent model in the case of a CRRA hedge fund manager. Our structural model naturally introduces manager's risk aversion through the channel of subjective valuation and thus provides a method to solve the difficulty in literature. Furthermore, all these works are established on the assumption of complete market. Since our model is structural in an incomplete market, the locked wealth in our model profoundly metamorphoses the model solution and thus differentiates our model from the previous works.

As for managerial ownership in hedge funds, Hodder and Jackwerth (2007) are the first to model managerial ownership in the hedge fund context, but their focus is not on managerial ownership per se and their model is in a discrete time setting. By calculating market values, LWY take the hedge fund managerial ownership into consideration as a model extension but is limited by their model settings: The managerial ownership is not tradable and thus idiosyncratic risk should have been modeled in the presence of managerial ownership. It is difficult to build a structural model that incorporates private portfolio choice, consumption and fund management simultaneously, especially in the presence of managerial ownership that translates into locked wealth in the private portfolio. More often than not, managerial ownership accounts for a large portion of the manager's total wealth. This large holding of the hedge fund shares complicates the private portfolio choice problem for the manager, creating a very complex interaction between her fund management and private wealth management. Despite the challenges, we successfully solve the structural model and thus contribute to literature.

[2]In this case, the idiosyncratic risk and market risk in the risky asset plays the same role and could be integrated into one shock as in PW.

[3]The conception that income flows are not perishable can be dated back to Hindy and Huang (1992 and 1993), Hindy, Huang, and Kreps (1992), which has been mentioned in PW.

Our model also contributes to the literature of portfolio choice in continuous time initiated by Merton (1969) and the literature of managerial hedging. In order to solve CEO compensation contract valuation problems, Ingersoll (2006) is the first to solve a continuous time constraint portfolio choice problem in an incomplete market when the CEO is assumed to lock a portion of wealth in the stock of the firm she manages. Our paper shares the methodology developed in Ingersoll (2006). Beyond the hedge fund valuation, our model allows us to contribute to a heated topic of whether a convex compensation causes a delegated portfolio manager's risk-shifting behavior. Carpenter (2000), Ross (2004), Basak, Pavlova, and Shapiro (2007) and Buraschi, Kosowski, and Sritakul (2014) have delved into this topic extensively. In our model, clearly, locked wealth, idiosyncratic risk (but not the market beta), and subjective valuation form an effective mechanism to alleviate the convex compensation effect of HWM in the hedge fund.

Last but not least, we realize that the locked wealth clause and the stochastic managerial ownership is in some sense unrealistic, since more often than not, the manager's private wealth is unobservable and managerial ownership is contracted as a constant. However, our model setting provides strong tractability that leads to closed form solutions to a complicated structural model and still generate rich model dynamics and implications. Importantly, our model helps us to understand managerial hedging behavior under fund management contexts which is new in literature.

The paper is arranged as follows: Section 2 presents our model economy, solves the manager's private portfolio choice problem given hedge fund operation, and delineates our valuation approach. Section 3 derives subjective valuation of the manager and the optimal leverage choice in the fund. In Section 4, we solve the investors' valuation and their participation constraint. We formally state the full equilibrium dynamics generated by the structural model, calibrate our model, report our findings, and explain these phenomena in Section 5. Section 6 concludes this paper.

2 Model Economy

2.1 Environment

Assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ admits a two-dimensional Brownian motion (B, Z) where B and Z are independent Wiener processes. Naturally define a filtration $\{\mathcal{F}_t : t \geq 0\}$ generated by (B, Z). In our model, B stands for shocks from the market wide systematic risk while Z refers to the idiosyncratic shocks from the unique asset holding in the hedge fund. The appearance of idiosyncratic risk indicates that our model economy lies in an incomplete market where the risk neutral measure is not well defined. Thus, throughout this paper, measure $\mathbb{P}$ denotes physical probability measure while operator $\mathbb{E}$ denotes mathematical expectation under $\mathbb{P}$. Further define space $\mathcal{L}^2_{[0,\infty)} \equiv$

$\left\{\chi : \int_0^t \chi_s^2 \mathrm{d}s < \infty, \forall t \in [0, \infty)\right\}$. Given this environment, there is a hedge fund managed by a manager as well as her management team and a group of investors who contract with the manager to make investment in the hedge fund at date 0.

2.2 Investment Opportunity

In our model economy, there is a risk-free bond with continuous return r and a market portfolio M whose price process is

$$\frac{\mathrm{d}M_t}{M_t} = m\mathrm{d}t + \sigma \mathrm{d}B_t, \tag{1}$$

with $M_0 > 0$, where m is the market instantaneous return, and σ is a positive constant representing the market volatility level. The risk-free bond and the market portfolio are traded without transaction cost in the market and thus are accessible to all investors as well as the hedge fund manager when she is managing her private portfolio.

In the hedge fund, however, the fund manager has relatively large opportunities to beat the market due to her unique and confidential investment skills which are unavailable to the investors. We assume that the fund manager is able to generate alpha return by trading some illiquid assets or constructing complicated portfolios but suffers from idiosyncratic risk, which is abstracted as a risky asset S with price process:

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \beta\sigma \mathrm{d}B_t + \epsilon \mathrm{d}Z_t, \tag{2}$$

and $S_0 > 0$. Clearly, β is the market loading of the risky asset, which can be completely hedged by the market portfolio. Although it is not necessary to assume that continuous time CAPM holds, but to simplify our exposition, we formally define $\alpha \equiv \mu - r - \beta(m - r)$ as the excess return of the risky asset; higher α means better investment skill of the manager. The last diffusion ϵ measures the magnitude of idiosyncratic risk of the risky asset, which can be interpreted as risk related to α. On one hand, due to the uniqueness of trading strategy implemented in hedge fund, the fund manager does not simply make passive investment in the market portfolio.[4] On the other hand, shares of hedge fund AUM can be neither traded nor replicated in the market, which contributes to market incompleteness.

2.3 Hedge Fund

Denote by W the AUM of the hedge fund. One should interpret W as the accounting asset measure or the amount of money that could be cashed if the fund were liquidated abruptly. When managing the

[4]We assume that the manager does not invest the hedge fund AUM into the market portfolio, since it is noticed in literature that hedge funds are significantly characterized by their investment style and special assets holdings; see e.g. Aragon and Nanda (2011).

fund, the fund manager invests a proportion π of W into the risky asset and the remaining $1-\pi$ into the risk-free bond. We do not confine π to be less than 1, since a π larger than 1 means the manager utilizes a leverage to augment her return in the alpha strategy. The only technical restriction on π is that $\pi W \in \mathcal{L}^2_{[0,\infty)}$ to exclude continuous time doubling strategy. Leverage is commonly used in hedge funds but sometimes the hedge fund manager may become cautious and hoard cash to escape from some extreme risk, such as financial crisis or sovereign debt crisis. The flexibility of the leverage under the manager's control is important to keep AUM stable and mitigate liquidation risks.

The investors' investment contract with the manager specifies managerial compensation as incentive fee which depends on HWM. We do not model management fee since 'in practice management fees are often intended to pay for the funds operating expenses (e.g., research)'.[5] Instead, we model a constant fraction c of W continuously paid as operation expenses, dividends, or withdrawal. In practice, HWM is a monotonically increasing process that is rearranged periodically. When AUM is below HWM, HWM is not affected by the evolution of AUM and thus is a deterministic process. The adjustment of HWM when AUM touches HWM depends on incentive fee contract. To model this, denote by H the process of HWM. If $W < H$, then H evolves deterministically as:

$$\frac{\mathrm{d}H_t}{H_t} = (r-c)\mathrm{d}t \tag{3}$$

with $H_0 = W_0$. The evolution of HWM means that the investors set the benchmark growth rate of the hedge fund HWM at the risk-free rate, but adjust it to the scale of AUM in consideration of deterministic outflows. When AUM touches HWM, H, we have $\mathrm{d}H_t > (r-c)H_t\mathrm{d}t$, and the manager collect a fraction k of the difference $\mathrm{d}H_t - (r-c)H_t\mathrm{d}t$, which contributes to a boundary condition in the manager's claim valuation problem.

Given leverage π, when $W < H$, AUM W evolves as:

$$\frac{\mathrm{d}W_t}{W_t} = \pi(\mu\mathrm{d}t + \beta\sigma\mathrm{d}B_t + \epsilon\mathrm{d}Z_t) + (1-\pi)r\mathrm{d}t - c\mathrm{d}t. \tag{4}$$

2.4 Manager's Private Wealth and Portfolio

The investors' investment contract stipulates that the hedge fund manager must lock a constant fraction ψ of her private wealth into the hedge fund as managerial ownership with intent to align the manager's incentive. Assume that the manager's private wealth process is X. Then at date 0, the manager contributes ψX_0 into the hedge fund asset and obtains a managerial ownership ϕ_0 such that $\psi X_0 = \phi_0 W_0$. After the commencement of the fund, the constant locked wealth clause is implemented by share transaction between the manager and the investors. For simplicity, we assume that both the

[5] See Drechsler (2014), page 2074.

manager and the investors should commit to the contract as long as the initial participation constraint that both sides' initial valuations in the fund are greater or at least equal to their initial investment amount is satisfied.

Under this setting, the managerial ownership ϕ is a stochastic process. The equation $\psi X_t = \phi_t W_t$ introduces a natural boundary for the validity of the constant locked wealth clause. When the hedge fund asset decreases or the manager's wealth increases, in order to maintain a constant proportion of wealth in the fund, the manager purchases shares of the fund measured by AUM accounting numeraire from the investors until $\phi = 1$, which means that the fund manager totally redeems the fund asset or equivalently the fund is entirely privatized. Define

$$\tau \equiv \inf\{t \geq 0 : \phi_t = 1\}. \tag{5}$$

At τ, the investment contract ceases. Before τ, we obtain the lower boundary for the manager's subjective valuation , that is ψX_t, or in normalized term,

$$b_t \equiv \frac{\psi X_t}{H_t}. \tag{6}$$

Given locked wealth in the hedge fund, the manager allocates her remaining wealth between the risk-free bond and the market portfolio. Assume that she invests proportion ω of her private wealth in the market portfolio such that $\omega X \in \mathcal{L}^2_{[0,\infty)}$, and the residual $1 - \psi - \omega$ proportion in the risk-free bond, then when $bH < W < H$, X evolves according to:

$$\frac{\mathrm{d}X_t}{X_t} = (1 - \omega - \psi)r\mathrm{d}t + \omega\frac{\mathrm{d}M_t}{M_t} + \psi\frac{\mathrm{d}W_t}{W_t} - \frac{C_t\mathrm{d}t}{X_t}, \tag{7}$$

where C is the manager's private consumption.

At any $t < \tau$, assume that $bH_t < W_t \leq H_t$. Define stopping time

$$\delta_t \equiv \inf\{s \geq t : W_s = H_s\} \tag{8}$$

as the next time after t when the manager collects her incentive fee. Further define

$$u_t \equiv \inf\{\tau, \delta_t\}, \tag{9}$$

then the manager whose relative risk aversion parameter is γ solves the following private portfolio choice problem:

$$J(X_t) = \sup_{C,\omega} \mathbb{E}_t\left[\int_t^{u_t} e^{-\rho(s-t)}\frac{C_s^{1-\gamma}}{1-\gamma}\mathrm{d}s + e^{-\rho(u_t-t)}J(X_{u_{t+}})\right], \tag{10}$$

subject to (7), where $X_{u_{t+}}$ denotes the right hand side limit of the private wealth process at the stopping time u_t, when the incentive fee has been collected if $u_t = \delta_t < \tau$. Notice that in problem (10), process π is assumed to be given but endogenized later. There are two major reasons why the hedge fund leverage level π is not solved here: Economically, the incentive fee is distributed among members of the management team, so the manager operates the hedge fund on behave of the management team to maximize claim valuation in the hedge fund by selecting optimal leverage there. Technically, under continuous time setting, the timing of HWM compensation is of zero measure on the time dimension. Besides, at each time when incentive fee is collected, the increment of wealth precess is of infinitesimal order. Thus, the HWM part cannot be clearly modeled in the utility objective. The implicit incentive of claim value maximization sways the manager's private portfolio and consumption from those in standard Merton's problem. We state the solution to the manager's private portfolio choice problem above in the following proposition.

Proposition 1. *Given leverage choice π in the hedge fund, when $bH < W < H$, the fund manager solves the private portfolio choice problem (10) as:*

$$J(X) = \zeta^{\gamma}\frac{X^{1-\gamma}}{1-\gamma}, \tag{11}$$

$$C_s = \frac{X_s}{\zeta}, \tag{12}$$

$$\omega = \frac{m-r}{\gamma\sigma^2} - \psi\pi\beta, \tag{13}$$

where ζ is defined as:

$$\zeta = \frac{\gamma}{(\gamma-1)\left(r + \frac{\rho}{\gamma-1} + \frac{1}{2}\frac{(m-r)^2}{\gamma\sigma^2} + \alpha\psi\pi - \psi c - \frac{1}{2}\gamma\psi^2\pi^2\epsilon^2\right)}. \tag{14}$$

Proof. To simplify derivation, we temporarily impose the restriction that π is a constant; the constant leverage in the hedge fund will be verified in the following section.

To solve (10), we derive the Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} 0 \quad = \quad & \sup_{C,\omega}\left\{\frac{C^{1-\gamma}}{1-\gamma} - \rho J(X) + \Big(\big(r + \omega(m-r) + \psi\pi(\mu - r) - \psi c\big)X - C\Big)J'(X)\right. \\ & \left. + \frac{1}{2}\big(\omega^2\sigma^2 + 2\omega\psi\pi\beta\sigma^2 + \psi^2\pi^2(\beta^2\sigma^2 + \epsilon^2)\big)X^2 J''(X)\right\}. \end{aligned} \tag{15}$$

The solution to this Merton's problem takes the form of (11). The first order conditions gives (12) and (13). By substituting the first order conditions into (15), we can derive ζ. □

Clearly, even though the hedge fund management is given in the manager's private portfolio choice problem, the manager's consumption and private wealth allocation is influenced by the hedge fund management through the channel of leverage π.

A word of caveat, however, is that the above proposition does not clearly states the solution to the manager's private portfolio choice when $W = H$, since under continuous time setting HWM adjusts instantaneously and the duration of this case is negligible over the time dimension[6]. One

[6] Formally, the Lebesgue measure of the set of time when $W = H$ is 0.

could interpret the manager's problem as she periodically solves problem (10) over the state space $bH < W < H$. As for the consumption, we assume that once the incentive fee is collected, it is added up into the manager's private wealth so that the constant consumption rate and thus the consumption smoothing effect are not affected.

2.5 The Valuation Approach

We now return to the hedge fund and expound the valuation techniques applied to the fund. As we have mentioned, the AUM process W represents the amount of money that could be drawn should the fund be liquidated. It disregards the future value increment in the fund due to the manager's capability to generate alpha. Thus, as suggested by GIR, agents in the hedge fund evaluate the fund not by its AUM, but by the valuation of their claims in the fund. A claim is something like a legal entitlement to (possessing) a particular payoff according to the terms in the investment contract and depending on some underlying asset. In addition, when discounting future cash flows in the hedge fund, different agents in the fund may use different discount rates: The investors are able to diversify their shares in the hedge fund by only allocating a limited portion of their wealth in the hedge fund, thus they use the market martingale pricing kernel to evaluate their claim; the manager takes a significant portion of stake in the hedge fund and cannot diversify the idiosyncratic risk in the hedge fund when making her private portfolio choice, so she needs to use her subjective martingale pricing kernel to discount her claim as is proposed in Ingersoll (2006)[7].

There is a large literature about claim valuation in finance and economics. Usually these works assume a complete market which implies the existence of a unique equivalent martingale measure under which all the cash flows of any claim can be discounted by the risk-free rate in the risk-neutral measure. In this case, any agent is able to align her marginal utility with a constant proportion of the unique state price almost surely. However, even in the complete market, the classic Black-Scholes formula is derived under a number of strict assumptions. Financial market in reality, on the other hand, may be always incomplete. Market incompleteness may result in many equivalent martingale measures when the market reaches the state of absence of arbitrage, making the previously mentioned pricing approach invalid and inviting subjective valuation. Because our model economy is incomplete, we part our way with economy in complete market by deriving martingale pricing (or valuation) kernels first and then use the equilibrium valuation approach to gain claim valuation, which is inspired by Merton (1976), Duffie (2001), GIR, and Ingersoll (2006).

[7] As a matter of fact, GIR have realized that subjective valuation should be applied in hedge fund context in the presence of manager's locked wealth in the hedge fund in their footnote 7 on page 1691. They also recognize that it is the working paper version of Ingersoll (2006) that first proposes subjective valuation.

Quite generally, the equilibrium valuation approach asserts that in the market equilibrium, agents in the market use some martingale pricing (or valuation) kernel to evaluate any claim or future cash flows: In a pure valuation context without control, the claim valuation discounted by the martingale pricing (or valuation) kernel should be a martingale. By definition, an agent's martingale pricing (or valuation) kernel is derived from that agent's marginal utility in their consumption by solving the portfolio choice problem and thus different martingale pricing (or valuation) kernels can be derived by different agents. In our model, as mentioned, the investors' claim valuation in the hedge fund are discounted by the market martingale pricing kernel Ξ which evolves as:

$$\frac{\mathrm{d}\Xi}{\Xi} = -r\mathrm{d}t - \frac{m-r}{\sigma}\mathrm{d}B_t, \tag{16}$$

and is derived from the marginal utility of the representative consumer in our model economy. However, due to her undiversifiable managerial ownership in the hedge fund, the fund manager evaluates her claim in the fund by virtue of her subjective martingale pricing kernel Θ that is derived in the next section.

3 The Subjective Valuation

3.1 The Manager's Claim Valuation

We first derive the manager's subjective martingale pricing kernel in the following proposition.

Proposition 2. *The manager's subjective martingale pricing kernel is defined as* $\Theta_t = e^{-\rho t}J'(X)$ *and evolves as:*

$$\frac{\mathrm{d}\Theta_t}{\Theta_t} = -(r + \alpha\psi\pi - \psi c - \gamma\psi^2\pi^2\epsilon^2)\mathrm{d}t - \frac{m-r}{\sigma}\mathrm{d}B_t - \gamma\psi\pi\epsilon\mathrm{d}Z_t, \tag{17}$$

with almost every $t \in [0, \tau)$.

Proof. Given J in (11), the dynamics of Θ is obtained by applying Itô's lemma. □

Compared with (16), the manager's subjective martingale pricing kernel above is divergent in two ways: The interest rate is represented by the subjective interest rate $r + \alpha\psi\pi - \psi c - \gamma\psi^2\pi^2\epsilon^2$, which is adjusted upward by the alpha return in the fund but rectified downward by the outflows and a certainty equivalent risk discount. The certainty equivalent risk discount reduces the manager's subjective interest rate since the locked wealth clause confines the manager's investment behavior and induces undiversified idiosyncratic risk for the manager. Besides, the idiosyncratic risk is priced in the subjective martingale pricing kernel whose valuation is proportion to the manager's risk aversion parameter, locked wealth, and the hedge fund's idiosyncratic risk.

Denote by $V(W,H)$ the manager's claim valuation with the following dynamics

$$\begin{aligned} \mathrm{d}V_t &= \Big(W_t\big(\pi(\mu-r)+r-c\big)V_W + H_t(r-c)V_H + \frac{1}{2}\pi^2W_t^2(\beta^2\sigma^2+\epsilon^2)V_{WW}\Big)\mathrm{d}t \\ &\quad +\pi W_t V_W(\beta\sigma\mathrm{d}B_t+\epsilon\mathrm{d}Z_t). \end{aligned} \tag{18}$$

Even though the manager's compensation includes only incentive fee, one should not treat her subjective valuation as only valuation of the incentive fee; instead, the managerial ownership and potential redemption of the fund asset are evaluated into V, leading to an interpretation of continuation value that to some extent is subtly different from present value interpretation in GIR, PW and LWY. Given her subjective martingale pricing kernel, the manager evaluates the product of this kernel and the total cum-dividend claim valuation as a super-martingale at any leverage level. In particular, she evaluates the product of her subjective martingale pricing kernel and the total cum-dividend claim value as a martingale at the optimal leverage level under her control. Formally, the pricing formula for V is derived from:

$$\sup_{\pi}\mathbb{E}\big[\mathrm{d}(\Theta V)\big]=0, \tag{19}$$

i.e.

$$\begin{aligned} 0 &= \sup_{\pi}\Big\{-(r+\alpha\psi\pi-\psi c-\gamma\psi^2\pi^2\epsilon^2)V \\ &\quad +(r+\alpha\pi-\gamma\psi\pi^2\epsilon^2-c)WV_W+(r-c)HV_H \\ &\quad +\frac{1}{2}\pi^2(\beta^2\sigma^2+\epsilon^2)W^2V_{WW}\Big\}, \end{aligned} \tag{20}$$

with boundary condition: $V(0,H)=0$ and $kV_W(H,H)-V_H(H,H)=k$. The reason why the first boundary is located at $W=0$ instead of $W=bH$ is that when $W=bH$, the manager totally privatize the hedge fund asset which could still generate potential returns for the manager should the economy persist. Furthermore, b ranges from 0 to 1, thus for the manager, the total feasible valuation interval constructs an augmented state space that ranges from 0 to H. The fact that the stochastic boundary b does not affect the dynamics in the V is the key to save managerial ownership ϕ as a state variable.[8]

To solve the Hamilton-Jacobi-Bellman equation (20), the second order condition

$$2\gamma\psi^2\epsilon^2V-2\gamma\psi\epsilon^2WV_W+(\beta^2\sigma^2+\epsilon^2)W^2V_{WW}<0 \tag{21}$$

must be satisfied. And we derive the optimal leverage as:

$$\pi^*=\frac{\alpha\psi V-\alpha WV_W}{2\gamma\psi^2\epsilon^2V-2\gamma\psi\epsilon^2WV_W+(\beta^2\sigma^2+\epsilon^2)W^2V_{WW}}. \tag{22}$$

[8]We realize that after privatization, the manager's problem is reduced into a standard Merton's problem with two risky assets which could be solved in closed form. Despite this, since the investment contract stops after privatization, the claim valuation approach becomes invalid. So, we are unable to identify the boundary by solving the Merton's problem. Our reasoning here justifies our approach by treading b as a stochastic boundary.

3.2 The Solution

An important characteristic of claims in hedge fund is homogeneity.[9] The reasoning is that the claim valuation must double when both HWM H and AUM W double. The implication of homogeneity in our model is that the state variable in effect is the ratio of W to H. Thus, as in GIR and LWY, homogeneity largely simplifies analysis by reducing the PDEs into ODEs.

Define $w \equiv \frac{W}{H}$ as the normalized AUM and $v(w) \equiv \frac{V(W,H)}{H} = V(w,1)$ as the normalized subjective valuation. Simple calculations show that on the augmented normalized state space $0 < w < 1$, v solves the following ODE

$$\begin{aligned}\big(\alpha\psi\pi^* - \gamma\psi^2\epsilon^2\pi^{*2} + (1-\psi)c\big)v(w) &= (\alpha\pi^* - \gamma\psi\epsilon^2\pi^{*2})wv'(w) \\ &\quad + \frac{1}{2}(\beta^2\sigma^2 + \epsilon^2)\pi^{*2}w^2v''(w), \end{aligned} \tag{23}$$

where

$$\pi^* = \frac{\alpha\psi v(w) - \alpha w v'(w)}{2\gamma\psi^2\epsilon^2 v(w) - 2\gamma\psi\epsilon^2 w v'(w) + (\beta^2\sigma^2 + \epsilon^2)w^2 v''(w)}, \tag{24}$$

with second order condition

$$2\gamma\psi^2\epsilon^2 v(w) - 2\gamma\psi\epsilon^2 w v'(w) + (\beta^2\sigma^2 + \epsilon^2)w^2 v''(w) < 0, \tag{25}$$

and boundary condition

$$v(1) = (1+k)v'(1) - k. \tag{26}$$

Note that another boundary condition $v(0) = 0$ is automatically satisfied if v is bounded. We state the solution to ODE (23) in the following proposition.

Proposition 3. *The normalized subjective valuation v takes the form of*

$$v(w) = \xi w^\eta, \tag{27}$$

where

$$\xi = \frac{k}{(1+k)\eta - 1}. \tag{28}$$

and η solves the following quadratic formula:

$$\begin{aligned} 0 &= \big(\alpha^2 + 2(1-\psi)c(\beta^2\sigma^2+\epsilon^2)\big)\eta^2 - \big(2\alpha^2\psi + 4(1-\psi)c\gamma\psi\epsilon^2 + 2(1-\psi)c(\beta^2\sigma^2+\epsilon^2)\big)\eta \\ &\quad + \big(\alpha^2\psi^2 + 4(1-\psi)c\gamma\psi^2\epsilon^2\big) \\ &\equiv f(\eta), \end{aligned} \tag{29}$$

which, under the parameter restriction that

$$f\left(\frac{1}{1+k}\right) < 0, \tag{30}$$

[9] In general, a function $\Upsilon(W,H)$ is called homogenous of degree one, if $\Upsilon(\kappa W, \kappa H) = \kappa\Upsilon(W,H)$ for any constant κ.

has a unique solution such that $v > 0$ and the second order condition is satisfied. The optimal leverage is a constant that is:

$$\pi^* = \frac{\alpha\psi - \alpha\eta}{2\gamma\psi^2\epsilon^2 - 2\gamma\psi\epsilon^2\eta + (\beta^2\sigma^2 + \epsilon^2)\eta(\eta - 1)}. \tag{31}$$

***Proof*.** The solution to ODE (23) takes the form of (27). Substitute (27) into the boundary condition (26), and we can identify parameter ξ. Substitute (27) into ODE (23), and we can obtain quadratic formula

$$-2(1-\psi)\big(2\gamma\psi^2\epsilon^2 - 2\gamma\psi\epsilon^2\eta + (\beta^2\sigma^2 + \epsilon^2)\eta(\eta - 1)\big) = \alpha^2(\psi - \eta)^2, \tag{32}$$

which is equivalent to $f(\eta) = 0$. Note that $f(0) > 0$, together with the parameter restriction (30), implies that $f(\eta) = 0$ has a unique solution such that

$$\eta > \frac{1}{1+k}, \tag{33}$$

and thus $v > 0$, which further implies that the second order condition

$$2\gamma\psi^2\epsilon^2 - 2\gamma\psi\epsilon^2\eta + (\beta^2\sigma^2 + \epsilon^2)\eta(\eta - 1) < 0 \tag{34}$$

holds by (32). Finally, π^* is derived by calculations. □

Interestingly, the functional form of the manager's subjective valuation coincides with that of a CRRA agent. However, the payment structure in the fund significantly affects the implicit risk aversion parameter for the manager form the subjective valuation.

Given the manager's subjective valuation, her participation constraint can be presented as:

$$v(1) \geq \phi_0, \tag{35}$$

which displays the second parameter restriction.

3.3 Decomposition of the Return on Subjective Valuation

Here, we calculate the return and risk premium of the manager's claim valuation. Denote by R the instantaneous return on the manager's subjective valuation V. By definition, R is derived as:

$$\begin{aligned} R\mathrm{d}t &\equiv \mathbb{E}\left[\frac{\mathrm{d}V}{V}\right] \\ &= \frac{W_t\big(\pi^*(\mu - r) + r - c\big)V_W + H_t(r-c)V_H + \frac{1}{2}\pi^{*2}W_t^2(\beta^2\sigma^2 + \epsilon^2)V_{WW}}{V}\mathrm{d}t. \end{aligned} \tag{36}$$

By homogeneity and ODE (23), we have

$$\begin{aligned} R &= \frac{\pi^*(\mu - r)wv'(w) + (r-c)v(w) + \frac{1}{2}\pi^{*2}(\beta^2\sigma^2 + \epsilon^2)w^2v''(w)}{v(w)} \\ &= (r + \alpha\psi\pi^* - \gamma\psi^2\epsilon^2\pi^{*2} - \psi c) + \eta\beta(m - r)\pi^* + \eta\gamma\psi\epsilon^2\pi^{*2}. \end{aligned} \tag{37}$$

Therefore, we decompose the instantaneous return on the manager's subjective valuation V into three parts: First, risk-free rate r should have been the base line of the return on the manager's subjective valuation. However, due to manager's risk aversion and subjective discount, the risk-free rate is increased by the alpha return in the hedge fund but lowered by the outflows in the fund and further lowered by a certainty equivalent adjustment $\gamma\psi^2\epsilon^2\pi^{*2}$, which means that the locked wealth serves as a constraint to reduce return in the manager's private portfolio choice due to the existence of idiosyncratic risk. Second, $\eta\beta(m-r)\pi^*$ represents a compensation for the systematic risk in the hedge fund. The market beta is augmented by leverage in the fund. The parameter, η, derived from the ratio $\frac{wv'(w)}{v(w)}$, measures marginal return on the manager's subjective valuation. We term this term the leveraged systematic risk premium. The third term, $\eta\gamma\psi\epsilon^2\pi^{*2}$, stands for a premium stemming from subjective valuation of the claim, which we dub as leveraged subjective risk premium.

4 Investors' Claim Valuation and Participation Constraint

In our model, the investors' investment contract is signed under full knowledge of the manager's private wealth and hedge fund operation, even though they can not interfere in the leverage choice in the hedge fund let along in the manager's private portfolio choice. Given π^*, in the region of $bH < W < H$, the investors' claim valuation $Y(W,H)$ evolves as:

$$\begin{aligned} \mathrm{d}Y_t &= \Big(W_t\big(\pi^*(\mu-r)+r-c\big)Y_W + H_t(r-c)Y_H + \frac{1}{2}\pi^{*2}W_t^2(\beta^2\sigma^2+\epsilon^2)Y_{WW}\Big)\mathrm{d}t \\ &\quad +\pi^*W_tY_W(\beta\sigma\mathrm{d}B_t+\epsilon\mathrm{d}Z_t). \end{aligned} \tag{38}$$

Similar to GIR, if we assume that ι fraction of cW is paid as dividends, where $\iota \in (0,1)$, then we have the investors' valuation equation:

$$\mathbb{E}\big[\mathrm{d}(\Xi Y) + \Xi(\iota cW)\mathrm{d}t\big] = 0, \tag{39}$$

i.e.

$$\begin{aligned} 0 &= -rY + (r+\alpha\pi^*-c)WY_W + (r-c)HY_H \\ &\quad +\frac{1}{2}\pi^{*2}(\beta^2\sigma^2+\epsilon^2)W^2Y_{WW} + \iota cW, \end{aligned} \tag{40}$$

with boundary conditions $Y(bH,H)=0$ and $kY_W(H,H)=Y_H(H,H)$. Note that b is stochastic and thus enters into the valuation Y as an implicit state variable.

Using the same homogeneity technique by defining $y(w) \equiv \frac{Y(W,H)}{H} = Y(w,1)$, we can reduce investors' valuation PDE (40) into the following ODE

$$cy(w) = \alpha\pi^*wy'(w) + \frac{1}{2}\pi^{*2}(\beta^2\sigma^2+\epsilon^2)w^2y''(w) + \iota cw, \tag{41}$$

with boundary condition

$$y(b) = 0, \tag{42}$$

and

$$y(1) = (1+k)y'(1), \tag{43}$$

when $b < w < 1$. The solution to investors' valuation is stated in the following proposition.

Proposition 4. *Given π^*, when $b < w < 1$, the investors' normalized valuation y takes the form of*

$$y(w) = l_1 w^{a_1} + l_2 w^{a_2} + \frac{\iota c}{c - \alpha\pi^*} w \tag{44}$$

where a_1 and a_2 solve the following quadratic formula

$$\frac{1}{2}\pi^{*2}(\beta^2\sigma^2 + \epsilon^2)a^2 + \left(\alpha\pi^* - \frac{1}{2}\pi^{*2}(\beta^2\sigma^2 + \epsilon^2)\right) a - c = 0, \tag{45}$$

and l_1 and l_2 solve the following linear system

$$\begin{pmatrix} 1-(1+k)a_1 & 1-(1+k)a_2 \\ b^{a_1} & b^{a_2} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} \frac{k\iota c}{c-\alpha\pi^*} \\ -\frac{\iota c b}{c-\alpha\pi^*} \end{pmatrix}. \tag{46}$$

***Proof*.** The proof follows from the solution to a standard non-homogeneous Cauchy-Euler ODE. □

The investors' participation constraint requires that $Y(W_0, H_0) \geq (1-\phi_0)W_0$ or equivalently,

$$y(1) \geq 1 - \phi_0, \tag{47}$$

which translates to the third parameter restriction.

5 Full Equilibrium Dynamics and Model Implication

5.1 Full Equilibrium Dynamics

It is necessary to summarize the full equilibrium dynamics in the structural model in the following theorem.

Theorem 1. *Under parameter restriction (30), the manager's participation constraint (35) and investors' participation constraint (47), given X_0 and $W_0 = H_0$, the fund manager who is contracted to lock a constant proportion ψ of her wealth in the hedge fund operates her private portfolio and the hedge fund asset simultaneously with the dual objectives (10) and (19). When $\psi X < W < H$, W evolves according to*

$$\frac{\mathrm{d}W_t}{W_t} = \pi^*(\mu \mathrm{d}t + \beta\sigma \mathrm{d}B_t + \epsilon \mathrm{d}Z_t) + (1-\pi^*)r\mathrm{d}t - c\mathrm{d}t, \tag{48}$$

and X evolves according to

$$\begin{aligned}\frac{\mathrm{d}X_t}{X_t} &= \frac{m-r}{\gamma\sigma^2}(m\mathrm{d}t+\sigma\mathrm{d}B_t)+\left(1-\frac{m-r}{\gamma\sigma^2}\right)r\mathrm{d}t \\ &\quad +\psi(\pi^*\alpha\mathrm{d}t+\pi^*\epsilon\mathrm{d}Z_t-c\mathrm{d}t) \\ &\quad -\frac{1}{\zeta}\mathrm{d}t, \end{aligned} \tag{49}$$

where π^* and ζ are defined in (31) and (14) respectively. When $W=H$, HWM adjusts instantaneously and the manager collects incentive fee. When $W=\psi X$, the hedge fund is totally privatized, and the investment contract ceases.

The dynamics of the manager's private wealth in (49) shows clearly the managerial hedging behavior: The first row is the standard Merton's optimal portfolio of a CRRA agent in a complete market. In the second row, however, due to the managerial ownership or the locked wealth, the manager benefits from leveraged alpha return but suffers from leveraged idiosyncratic risk from the fund asset.

5.2 Model Calibration and Simulation Experiment

We refer to a number of previous works to determine our benchmark parameter values, which are used to present intuitive illustration and interpretation about our model. Table 1 summarizes key variables in this paper while Table 2 presents values of the benchmark parameters. We refer to Wachter (2013) to set market environment parameters r, m, and σ so that market Sharpe ratio is 40%. Incentive fee fraction k is set at 20% and ratio of outflows c at 2% of which half is paid as dividends (ι), which are standard in hedge fund literature. We select $\alpha = 1.2\%$, compatible with that in LWY and also consistent with empirical findings in Buraschi, Kosowski, and Sritrakul (2014). In a series of papers to construct hedge fund return factors, Fung and Hsieh (1997, 2002 and 2004) claim that hedge fund returns are not highly related to traditional systematic or market factors but are correlated to common factors special to hedge fund treading strategies. Thus, we choose ϵ equal to 5% and β equal to 0.2 so that the unleveraged idiosyncratic volatility is twice of the unleveraged market exposure in risky asset S. We consider a hedge fund whose initial asset size is 100 and a manager whose initial wealth is 10. By assuming that the locked wealth is 70%, we derive an initial managerial ownership of 7%, which is almost the average managerial ownership reported in Agarwal, Daniel, and Naik (2009). Lastly, for simplicity, we assume that $\rho = r$.

Table 1 about here

Table 2 about here

Then we check parameter restriction (30), the manager's participation constraint (35), and investors' participation constraint (47) and find that they are satisfied. Figure 1 plots the normalized valuations of the manager's and the investors' claims. In the left hand side panel, the manager's normalized subjective valuation increases with the normalized state variable, and increases at an increasing rate. The convex valuation seems to suggest an implicit risk preference of the manager when operating the fund given our benchmark parameters, which contrasts with the results in PW and LWY that a risk neutral manager displays implicit risk averse in her valuation. However, from the second order condition (25) and the definition of leverage in equation (24), we point out that the concavity per se dose not portray the manager's risk taking behavior as it does in PW and LWY and thus may not be interpreted as implicit risk aversion in our model. Besides, the fact that convex subjective valuation does not necessarily hold under other parameter settings due to the form of valuation function (27) evidences the significant influence of the investment contract on the manager's claim valuation. Given the concavity of subjective valuation, by referring to (28), we know that the subjective valuation uniformly increases over the augmented normalized state space with respect to incentive fee fraction k, which differs from the corresponding result in PW. Furthermore, at date 0, the manager's initial normalized valuation in the fund is 0.31, much larger than her managerial ownership amount 0.07, showing that managemnet claim in the fund carries substantial potential return for her. According to decomposition (37), we further calculate that the return on the manager's subjective valuation is 9.18%, which more than doubles the subjective interest rate 4.44%. The leveraged systematic risk premium is 2.68% while the leveraged subjective risk premium is 2.05%, both of which are of considerable economical significance.

Figure 1 about here

The right hand side panel of Figure 1 shows the market valuation of the investors' claim over the initial state space should the claim be traded freely at the commencement of the fund. In particular, the initial normalized valuation for the investors is 1.71, much larger than their normalized money amount 0.93 that is invested, which corroborates their participating in the investment contract. On the normalized state space, the investors' valuation increases with the normalized asset in the fund, since the source of inefficiency for the investors lies in the complete redemption of the fund asset by the manager due to the locked wealth clause. Intuitively, redemption deprives the investors of the opportunity to benefit from future valuation creation and dividends in the fund asset. Typically, in our setting, a concave valuation function manifests implicit risk aversion of the investors.

In order to obtain further insight into our full equilibrium dynamics, we conduct simulation experiment on our model economy. Our simulation lasts for 5 years, and in each year 250 trading days are assumed. Thus, consistent with continuous time econometrics simulation literature, e.g. Aït-Sahalia (2002), we choose step length of $\frac{1}{250}$ to imitate daily sample paths. In Figure 2, we plot one set of realized paths in the full equilibrium dynamics in our simulation experiment. The top panel shows that the hedge fund AUM stays far away from the redemption boundary (the dashed line) and is well bounded by the HWM (the dotted line). Since the fund asset and the manager's wealth share the same systematic shock, the manager's wealth path exhibits a similar trend to AUM, which is plotted in the second panel. The middle panel marks the incentive fee collected by the manager at trading days when the closing AUM exceeds the closing HWM one day ago; in these cases, the manager garners 20% of the difference between the two and the closing HWM adjusts correspondingly. The fourth panel plots the stochastic managerial ownership caused by the locked wealth clause. By Itô's lemma, it is easy to show that in the full equilibrium dynamics, managerial ownership ϕ follows a geometric Brownian motion. And in our simulation experiment, managerial ownership seems to be stable, indicating that the drift term in ϕ is in effect trivial. Last but not least, we plot the manager's subjective valuation which is appreciably larger than her wealth and generally increases with AUM in the fund.

Figure 2 about here

5.3 Hedge Fund Leverage

Hedge fund leverage, representing the manager's risk taking incentives, is of vital importance in our model. Comfortably, this set of parameters generate an optimal leverage π^* of 1.48, approximate the average long only leverage of 1.36 reported in Ang, Gorovyy, and van Inwegen (2011). Furthermore, since leverage in our model is endogenously related to the manager's private portfolio parameters, we are interested in how these parameters affect the leverage decision. Figure 3 plots the comparative static analyses conducted on hedge fund leverage. First, the northwest panel shows that optimal leverage surges when the contracted locked wealth decreases, showing strong incentive alignment effect in our model: The more the manager's stake in the fund, the lower risk she is willing to take in the hedge fund. Since managerial ownership is positively correlated with locked wealth given wealth and AUM, our model supports the empirical finding in Aragon and Nanda (2011) that managerial risk taking is less if the managerial ownership is higher. On the other hand, due to the implicit risk aversion observed in the investors' valuation, these investors essentially detest high leverage and thus demand sizable locked wealth to reach the investment contract. We further find that the minimum locked wealth required to meet the investors' initial participation constraint is 48.28% in our model, signifying the importance of managerial ownership in delegated portfolio management problems.

Second, we find that idiosyncratic risk negatively affect leverage in the fund, even though the subjective valuation suggests a risk preference behavior for the manager, as is plotted in the northeast panel of Figure 3. The key to understand this is to recall the fact that the subjective valuation is a channel through which the manager transfers her aversion to idiosyncratic volatility in her private portfolio to the hedge fund operation. Since idiosyncratic risk cannot be diversified in both the hedge fund and her private portfolio, the manager always attempts to maintain a reasonable level of the idiosyncratic volatility. A natural conjecture is that the effect of market loading would be negatively correlated with optimal leverage. This surmise, however, is refuted by the southwest panel in Figure 3 that shows a seemingly counter-intuitive increasing leverage with respect to systematic risk. Actually, optimal leverage behavior in reaction to market beta demonstrates the role of managerial hedging: The manager is actually attached to systematic risk when managing the fund and the managerial ownership renders her a chance to hedge the market risk in the fund so that she can enjoy the alpha return better. This managerial hedging effect is unique in our model. The different effects of the two risk components on leverage can be tested in the following empirical model:

$$\pi_i = \text{Constant} + B_1\epsilon_i + B_2\beta_i + \text{Control Variables} + e_i, \tag{50}$$

where π_i is the cross-sectional hedge fund leverage, ϵ_i and β_i are cross-sectional idiosyncratic risk and market exposure respectively estimated from hedge fund AUM, and e_i stands for error term. If the regression is run in a sample of funds with managerial ownership, our model predicts that B_1 should be significantly negative while B_2 should be significantly positive.

Finally, we consider the effect of risk aversion parameter on optimal leverage, which is difficult to examine directly in previous models. Quite intuitively, through the subjective valuation channel, optimal leverage decreases in the manager's risk aversion parameter: A more risk averse manager behaves more cautious in the hedge fund.

Figure 3 about here

We also perform comparative static analyses on other parameters. Typically, consistent with previous works, hedge fund leverage increases with the excess return of the risky asset. However, different from e.g. Guasoni and Obłój (2013), incentive fee fraction does not affect the optimal leverage. Intuitively, any agent's risk taking incentive is determined by the concavity of her valuation function. And in our model, the form of subjective valuation (26) exhibits a separation between magnitude and concavity. Incentive fee affects valuation in respect of magnitude instead of concavity. Thus, leverage in our model is irrelevant to incentive fee. It is then tempting to think that incentive fee as a unique income source for the manager does not affect the manager's consumption. Although by (14) consumption rate is not affected by k, total consumption can still be affected by incentive payment since that payment directly increases the manager's private wealth.

6 Conclusion

Since the seminar work of GIR, modeling hedge funds has been an emerging and developing research field in finance. Our paper contributes to this research field by solving an elegant structural model in an incomplete market that incorporates the manager's private portfolio and consumption choice, hedge fund valuation and operation, and investors' participation constraint conjointly. In our model, the manager's private wealth is linked to the hedge fund primarily by an investment contract that specifies a constant portion of her private wealth invested in the hedge fund asset. Through the equilibrium valuation approach, we save the stochastic managerial ownership as an intractable state variable and derive very explicit closed form solutions that are easily analyzed. As mentioned, there are many interesting findings in our unique model settings. In particular, the systematic risk and idiosyncratic risk play very different roles in affecting the optimal leverage of the hedge fund, indicating the importance of taking managerial hedging into consideration when constructing investment contracts.

For future studies, it is an important topic to analyze compensation contracts in hedge funds from the perspective of contract design so as to explain why some characteristics are unique to hedge funds such as HWM. Agency problems are also necessary to be broached into the hedge fund context. For instance, we could have modeled that the manager exerts effort in hedge fund alpha return but suffers from a cost in her private portfolio choice problem; we leave this extension to future works. Lastly, we hope that our work can provide some important and useful guidance for empirical research of hedge funds.

References

Aït-Sahalia, Y., 2002, Maximum likelihood estimation of discretely sampled diffusions: A closed-form approximation approach, *Econometrica* 70(1), 223-262.

Agarwal, V., N. Daniel, and N. Naik, 2009, Role of managerial incentives and discretion in hedge fund performance, *Journal of Finance* 64(5), 2221-2256.

Ang, A., S. Gorovyy, and G. van Inwegen, 2010, Hedge fund leverage, *Journal of Financial Economics* 102(1), 102-126.

Aragon, G., and V. Nanda, 2011, Tournament behavior in hedge funds: High-water marks, fund liquidation, and managerial stake, *Review of Financial Studies* 25(3): 937-974.

Basak, S., A. Pavlova, and A. Shapiro, 2007, Optimal asset allocation and risk shifting in money management, *Review of Financial Studies* 20(5): 1583-1621.

Buraschi, A., R. Kosowski, and W. Sritakul, 2014, Incentives and endogenous risk taking: A structural view of hedge fund alphas, *Journal of Finance* 69(6), 2819-2870.

Carpenter, J., 2000, Does option compensation increase managerial risk appetite? *Journal of Finance* 55(5), 2311-2331.

Drechsler, I., 2014, Risk choice under high-water marks.*Review of Financial Studies* 27(7), 2052-2096.

Duffie, D., 2001, Dynamic asset pricing theory, Princeton University Press, Princeton, NJ.

Fung, W., and D. Hsieh, 1997, Empirical characteristics of dynamic trading strategies: The case of hedge funds, *Review of Financial Studies* 10(2), 275?302.

Fung, W., and D. Hsieh, 2002, Asset-based style factors for hedge funds, *Financial Analyst Journal* 58(5), 16-27.

Fung, W., and D. Hsieh, 2004, Hedge fund benchmarks: A risk based approach, *Financial Analyst Journal* 60(5), 65-80.

Goetzmann, W., J. Ingersoll, and S. Ross, 2003, High-water marks and hedge fund management contracts, *Journal of Finance* 58(4), 1685-1718.

Guasoni, P. and J. Obłój, 2013, The incentives of hedge fund fees and high-water marks, *Mathematical Finance*, forthcoming.

Hindy, A., and C. Huang, 1992, Intertemporal preferences for uncertain consumption: A continuous time approach, *Econometrica* 60(4), 781-801.

Hindy, A., and C. Huang, 1993, Optimal consumption and portfolio rules with durability and local substitution, *Econometrica* 61(1), 85-121.

Hindy, A., C. Huang, and D. Kreps, 1992, On intertemporal preferences in continuous time: the case of certainty, *Journal of Mathematical Economics* 21(5), 401-440.

Hodder, J., and J. Jackwerth, 2007, Incentive contracts and hedge fund management, *Journal of Financial and Quantitative Analysis* 42(4), 811-826.

Ingersoll, J., 2006, The subjective and objective evaluation of incentive stock options, *Journal of Business* 79(2), 453-487.

Lan, Y., N. Wang, and J. Yang, 2013, The economics of hedge funds, *Journal of Financial Economics* 110(2), 300-323.

Merton, R., 1969, Lifetime portfolio selection under uncertainty: The continuous time case, *The Review of Economics and Statistics* 51(3), 247-257.

Merton, R., 1976, Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics* 3(1), 125-144.

Panageas, S., and M. Westerfield, 2009, High-water marks: High risk appetites? Convex compensation, long horizons, and portfolio choice, *Journal of Finance* 64(1), 1-36.

Ross, S., 2004, Compensation, incentives, and the duality of risk aversion and riskiness, *Journal of Finance* 59(1), 207-225.

Wachter, J., 2013, Can time-varying risk of rare disasters explain aggregate stock market volatility? *Journal of Finance* 68(3), 987-1035.

TABLE 1

Summary of Key Variables

Variable	Symbol
Price Process of Market Portfolio	M
Price Process of Unleveraged Risky Asset	S
Manager's Private Wealth	X
Manager's Consumption	C
Inverse of Manager's Consumption Rate	ζ
Total Asset under Management (AUM)	W
High-Water Mark (HWM)	H
Market Martingale Pricing Kernel	Ξ
Subjective Martingale Pricing Kernel	Θ
Market Valuation of the Investors' Claim	Y
Subjective Valuation of the Manager's Claim	V
Normalized AUM	w
Normalized Valuation Boundary	b
Managerial Ownership	ϕ
Normalized Market Valuation of the Investors' Claim	y
Normalized Subjective Valuation of the Manager's Claim	v
Leverage Choice	π
Optimal Leverage Choice	π^*

This table summarized key variables in this paper.

TABLE 2

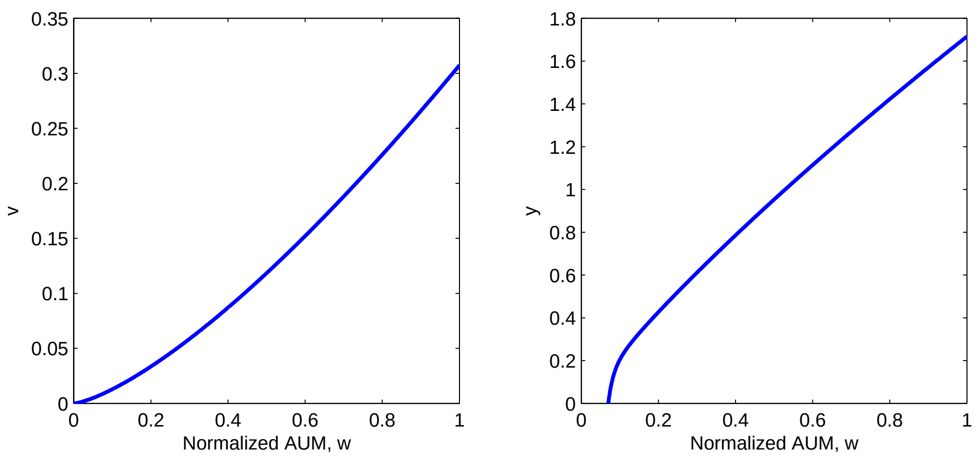
Summary of Key Parameters and Parameter Calibration

Variable	Symbol	Benchmark Value or Identification
Risk-Free Rate	r	0.05
Return on Market Portfolio	m	0.1
Volatility of Market Portfolio	σ	0.125
Manager's Utility Discount Rate	ρ	0.05
Manager's Risk Aversion Parameter	γ	3
Unleveraged Excess Return	α	0.012
Loading of the Risky Asset on Market	β	0.2
Return on the Risky Asset	μ	0.072
Idiosyncratic Volatility	ϵ	0.05
Expenses, Dividends or Withdrawal Rate	c	0.02
Fraction of Dividends	ι	0.5
Incentive Fee Rate	k	0.2
Proportion of Locked Wealth	ψ	0.7
Manager's Initial Wealth	X_0	10
Initial AUM	W_0	100
Initial HWM	H_0	100
Initial Managerial Ownership	ϕ_0	0.07

This table summaries key parameters, along with their values or identification, in this paper.

FIGURE 1

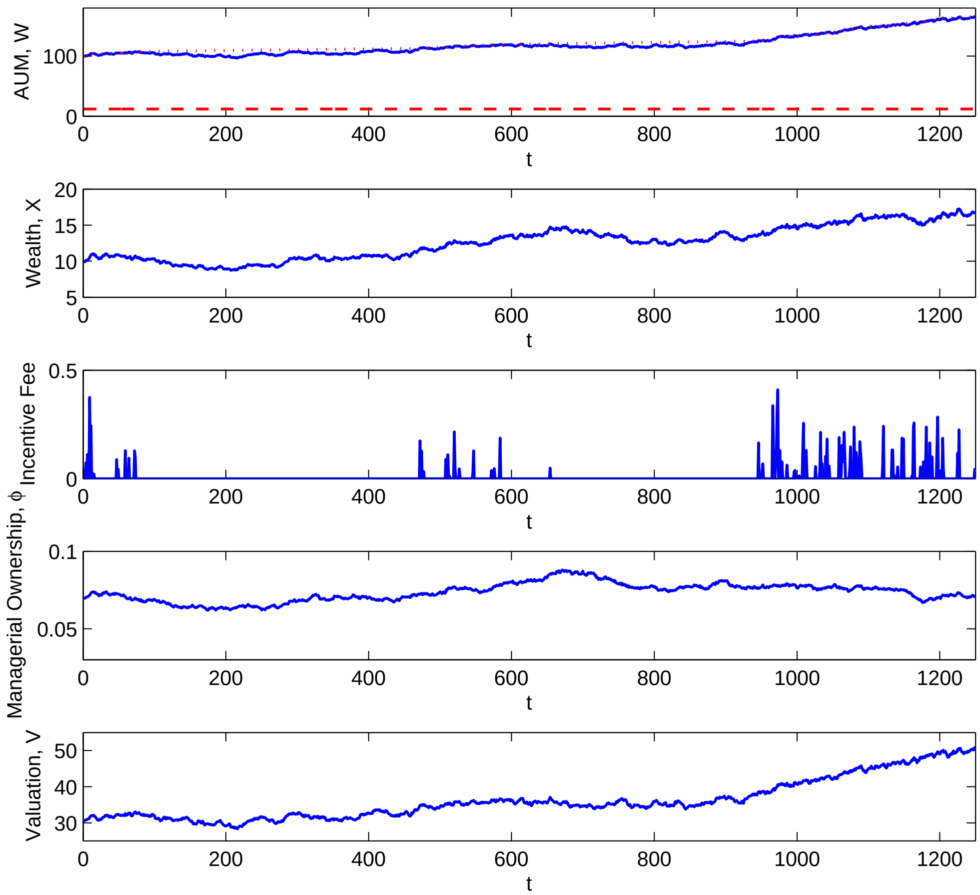
Normalized Valuations



This figure plots normalized subjective valuation of the manager's claim v on the augmented normalized state space in the left hand side panel and normalized market valuation of the investors' claim y on the initial state space in the right hand side panel. The parameters are set at values in Table 2.

FIGURE 2

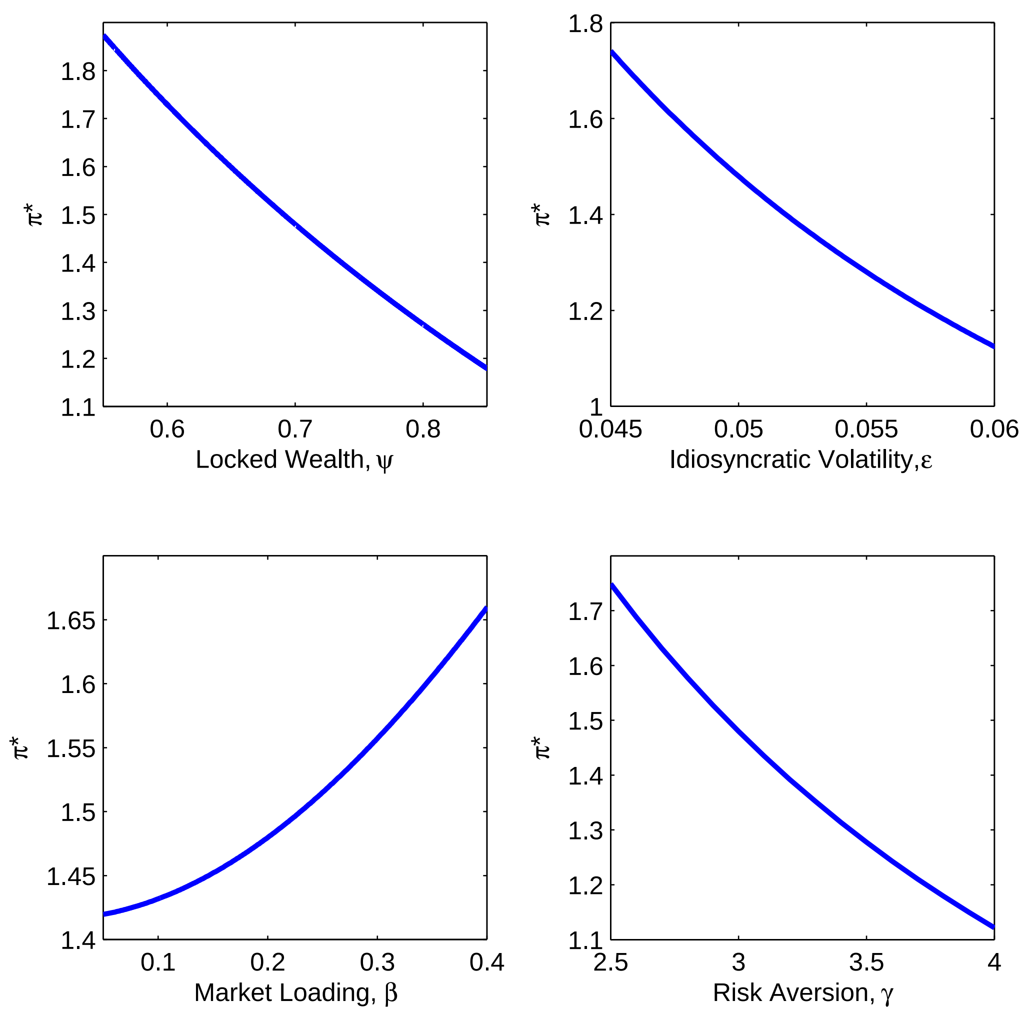
Simulated Path



This figure plots one set of sample paths generated in the full equilibrium dynamics. This simulation lasts for 5 years and in each year 250 trading days are assumed. The parameters are set at values in Table 2. The top panel plots the dynamics of AUM W (the solid line), which is bounded by HWM H (the dotted line) and a redemption boundary bH (the dashed line). The second panel plots the evolution of the manager's private wealth X. In the middle panel, we mark the dates when the manager collects her incentive fee. The fourth panel plots the dynamics of stochastic managerial ownership induced by locked wealth clause. The last panel plots the manager's subjective valuation.

FIGURE 3

Comparative Static Analyses on Optimal Leverage



This figure plots results of comparative static analyses performed on optimal leverage π^* with respect to locked wealth ψ, idiosyncratic volatility ϵ, market loading on risk asset β, and the manager's risk aversion parameter γ.