Optimal Contracts in a Continuous-Time Delegated Portfolio Management Problem

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This article studies the contracting problem between an individual investor and a professional portfolio manager in a continuous-time principal-agent framework. Optimal contracts are obtained in closed form. These contracts are of a symmetric form and suggest that a portfolio manager should receive a fixed fee, a fraction of the total assets under management, plus a bonus or a penalty depending upon the portfolio's excess return relative to a benchmark portfolio. The appropriate benchmark portfolio is an active index that contains risky assets where the number of shares invested in each asset can vary over time, rather than a passive index in which the number of shares invested in each asset remains constant over time.

Fund managers' compensation schemes in a delegated portfolio management setting have been of special interest to both academics and practitioners. Academic interest stems in part from the rapid growth of managed funds over the last two decades. Gompers and Metrick (1998) report that by December 1996, mutual funds, pension funds, and other financial intermediaries held discretional control over more than half of the U.S. equity market. It is thus of importance to study the impact of institutional trading on asset prices and to integrate into one model both asset pricing and delegated portfolio management compensation, as advocated by Brennan (1993), Allen and Santomero (1997), and Allen (2001). It is also important to understand the relationship between a manager's trading strategies and his implicit incentives or career concerns [see, e.g., Brown, Harlow, and Starks (1996), Brown, Goetzmann, and Park (1997), Chevalier and Ellison (1999), and Chen and Pennacchi (2000)]. These concerns include internal promotion from a small to a large fund, higher outside offers, and the flow of money through the fund under management.[1] The importance of fund manager compensation is also underscored by Congressional passage in 1970 of the Amendment to

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[1] For example, Chevalier and Ellison (1999) find that due to mutual fund managers' career concerns, younger managers take on lower unsystematic risk and deviate less from typical behavior than their older counterparts.

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the Investment Advisors Act of 1940 (hereafter "the Amendment"), which limits the types of compensation schemes that may be used by mutual funds, pension funds, and other publicly registered investment companies. Of particular interest is the requirement that when performance-based incentive fees involving a benchmark are used, then they must be symmetric around the chosen benchmark. Thus if an investment company is allowed to receive a bonus when its portfolio return is above the return on the benchmark portfolio, then it must also receive a penalty when its portfolio return is below the return on the benchmark portfolio.[2] Although performance-based compensation schemes have become quite popular in recent years, there is considerable debate among academics over whether the symmetric incentive fee now required by law is in fact economically justified based on modern financial economics.[3]

In this article we derive optimal portfolio management contracts in closed form in a relatively simple economy. Previous analyses of the principal-agent problem have not attempted to produce an optimal contract from a contract space that covers the symmetric incentive fees. Moreover, due to the absence of the stock prices in prior models, benchmark portfolios, such as a "passive index," in which the number of shares invested in each asset remains fixed over time, are not captured in their contract space. As a result, it has not been possible, using prior models, to examine the fundamental question of whether the symmetric compensation scheme prescribed by the Amendment for fund managers is efficient in a general principal-agent framework. Furthermore, it has not been possible to more thoroughly address issues regarding the impact of delegated portfolio management on equilibrium asset prices, nor the impact of fund managers' career concerns on their trading strategies.

We thus attempt to provide an economic foundation that may be used for the assessment of the symmetric incentive fees, like those required by the Amendment, for the study of the impact of managers' career concerns on their trading behavior and for the development of asset pricing models in the presence of delegated portfolio management. To do so, we analyze the relationship between an individual investor and a professional portfolio manager in a continuous-time principal-agent framework. The investor (principal) entrusts her funds to and provides a contract for the manager (agent). In our optimal contract solution, both the principal's and the agent's dynamic maximization problems are solved simultaneously: the agent's portfolio choice

[2] Prior to 1970, asymmetric incentive fees were also used. Under this fee structure, the management would receive a bonus if its portfolio outperformed the benchmark portfolio, and no penalty even if its portfolio underperformed the benchmark. Notice that the Amendment does not require funds to use any benchmarks. It only requires that if a fund uses an incentive fee involving a benchmark, the fee must be symmetric around the benchmark.

[3] For detailed discussion on mutual fund fee structure, see, for example, Starks (1987), Grinblatt and Titman (1989), Golec (1992), Brown and Goetzman (1995), Das and Sundaram (1998), and Cuoco and Kaniel (2000). For compensation schemes for hedge fund managers, see, for example, Fung and Hsieh (1997), Goetzmann, Ingersoll, and Ross (1997), and Ackermann, McEnally, and Ravenscraft (2000).

depends upon the contract assigned by the principal, and the principal's contract takes into account the agent's portfolio choice.

Our optimal contracts are obtained from a large contract space and they are shown to be of a symmetric form. They suggest that a fund manager should be paid a fixed fee, a fraction of the total assets under management, plus a bonus or a penalty depending upon the excess return on the portfolio. We also show that the appropriate benchmark is an "active index" in which the number of shares invested in each asset changes over time, rather than the passive index. For tractability of the model, we assume that the manager's utility function is of a negative exponential form. When the investor also has an exponential utility function, optimal contracts are obtained in closed form using a class of cost functions. If instead a general utility function for the investor is adopted, then the investor's maximization problem becomes analytically intractable in the presence of a general cost function. To provide insight into our model in that case, we treat a special case in which the manager's cost function is a constant. Optimal contracts are then derived in closed form.

Our model can be interpreted as a standard principal-agent model in which the agent's action affects both the drift (i.e., the return) and the diffusion rates (i.e., the risk) simultaneously, thus representing an extension of the previous principal-agent literature. Our analyses of the portfolio management problem show that the manager's action or his portfolio policy appears in both the drift and the diffusion terms. Previous continuous-time principal-agent models such as the well-known Holmström and Milgrom (1987), Schättler and Sung (1993), and Sung (1995) models cannot be applied to the delegated portfolio management problem because they do not allow the agent's action to influence both the drift and the diffusion terms simultaneously.[4] In Holmström and Milgrom and Schättler and Sung, for example, the agent controls the drift rate alone. And in Sung, the agent controls the drift and diffusion rates separately, and there are two independent control variables, so that the diffusion rate is held constant when the agent controls the drift rate, and vice versa.

The rest of the article is organized as follows. We briefly review the related theoretical literature on the delegated portfolio problem in the next section. Section 2 describes the basic setup. Section 3 derives an expression for the optimal compensation in terms of the agent's value function. In Sections 4 and 5, two types of problems are treated and optimal contracts are obtained. Some concluding remarks regarding the model are offered in Section 6. Appendix A shows that under a regularity condition, the Bellman equation is both a necessary and a sufficient condition for a dynamic maximization problem. Proofs of Proposition 1, of Theorem 1 and Corollary 1, and of Proposition 2 and Theorem 2 are presented in appendixes B, C, and D, respectively.

[4] Bolton and Harris (1997) and Detemple, Govindaraj, and Loewenstein (2001) extend the Holmström–Milgrom model to include more general output processes.

1. Related Literature

In this section we briefly review the theoretical research on the delegated portfolio management problem, which falls into three categories and typically takes contract forms as exogenously given.

The first category tries to address the implications of an appropriate benchmark within the symmetric contract form prescribed by the Amendment. Within a linear and symmetric contract form, Admati and Pfleiderer (1997) show in a one-period setting that the use of a benchmark portfolio of risky assets such as an index fund cannot be easily rationalized. Since their objective is not to study optimal contracts, no alternative benchmarks are offered. The Admati–Pfleiderer result challenges the common practice where portfolios of risky assets are often adopted as benchmarks for incentive fees and, as a result, it bears potentially significant policy implications. However, it is important to examine the issue regarding appropriate benchmarks in a multiperiod relationship as studied in this article, because this enlarges the space for a benchmark portfolio of risky assets over what is possible in a one-period model. The larger space includes not only a passive index but also an active index. Notice that the active index cannot be captured in a one-period model in which the manager invests only once.[5]

The second category of delegated portfolio management research compares the symmetric and asymmetric incentive fee structures. For example, Starks (1987) analyzes and compares the symmetric and asymmetric contracts and shows that the symmetric contract dominates the asymmetric one in inducing the manager to choose the portfolio policy desired by the investor. In a one-period and three-state risk-sharing model, Das and Sundaram (1998), however, find little justification for the symmetric contract and develop conditions under which the asymmetric contract provides a Pareto-dominant outcome. Stoughton (1993) examines the impact of the linear and quadratic contracts on both the investor's welfare and the manager's effort to acquire information. He shows in a limiting case that the quadratic contract may allow the investor to achieve the first-best result in a moral hazard model.

The third category of research examines the impact of given symmetric and asymmetric fee structures on the manager's portfolio policy and asset prices in an equilibrium setting. For example, on one hand, Grinblatt and Titman (1989) show that given an option-type asymmetric contract, if the manager can hedge his compensation, then he would choose investment strategies that increase the fund's volatility. On the other hand, Carpenter (2000) shows that if the manager cannot hedge, then the option compensation may not necessarily lead to greater risk seeking. Assuming that the manager's compensation

[5] Previous analyses of the agency relationship between an investor and a manager have assumed a discrete-time (often one-period) relationship. See, for example, Bhattacharya and Pfleiderer (1985), Dybvig and Spatt (1986), Cohen and Starks (1988), Kihlstrom (1988), Allen (1990), Stoughton (1993), Heinkel and Stoughton (1994), Dow and Gorton (1997), Lynch and Musto (1997), Garcia (2000a), Jiang (2000), Wonder (2000), and Core and Qian (2001). See also Mamaysky and Spiegel (2001) for a different approach to the issue of mutual funds.

is based on performance relative to an index, Brennan (1993) examines the expected stock returns in equilibrium for a one-period mean-variance economy and finds that the choice of the benchmark portfolio affects the equilibrium structure of expected returns. Cuoco and Kaniel (2000) analyze the implications of a compensation scheme that includes both the symmetric and the asymmetric structures. They find that symmetric performance fees induce a significant positive effect on the equilibrium prices of stocks included in the benchmark portfolio, a significant negative effect on their equilibrium Sharpe ratios, and a marginally positive effect on their equilibrium volatilities. They also find that asymmetric fees can generate the opposite pricing implications.

2. The Basic Setup

Consider a simple economy in which a principal hires an agent to manage her portfolio. For example, the principal may represent all the investors of a mutual or pension fund, and the agent then represents the fund company.[6] The time horizon is taken to be $[0, T]$. The principal invests $W(0)$ with the agent at $t = 0$, and the value of this investment is $W(T)$ at $t = T$. For tractability, we assume that the principal cannot withdraw funds from or add funds to the portfolio. In the case of a pension fund, early withdrawals incur heavy penalties and new contributions typically take place on certain dates, such as paydays. In the case of a mutual fund, this assumption holds well for a one-day contract period because the fund inflows and outflows occur only at the end of the day. For a longer contract period, this assumption implies that the fund inflows offset the fund outflows during the contract period or that there are no net fund flows.

There is one riskless bond and N risky stocks available for the agent to trade at any time between 0 and T. Assume that the price $\mathcal{B}(t)$ of the riskless bond follows a deterministic process $d\mathcal{B}(t) = r\mathcal{B}(t)\,dt$ and that the interest rate r is a constant. Also assume that the price process $P_i(t)$ for each risky stock that pays no dividends is described by the following geometric Brownian motion:

$$dP_i(t) = P_i(t)\mu_i\,dt + P_i(t)\sigma_i\,dB_t,$$

where μ_i is a constant, σ_i is the ith row of a constant matrix σ in $R^{N\times d}$ with linearly independent rows, and the transpose of B, $B^T \equiv (B_1, \ldots, B_d)$, $d \geq N$, is a standard Brownian motion defined on a complete probability space $(\Omega, \mathrm{P}, \mathcal{F})$. Each $\omega \in \Omega$ specifies a complete history of the Brownian motion. We shall write the N stock price processes in a compact form as $dP(t) = \mathrm{diag}(P)[\mu\,dt + \sigma\,dB_t]$, where $P(t)$ and μ are two $(N \times 1)$ vectors

[6] Here we have omitted the potential conflicit between the fund company and the manager hired by the company to actively manage the fund. We shall use principal (agent) and investor (manager), and fund company and manager interchangeably.

and where $\mathrm{diag}(P)$ represents an $(N \times N)$ matrix with the diagonal terms being $P_1(t), P_2(t), \ldots, P_N(t)$, and the off-diagonal terms being zero.

It is well known that the wealth process $W(t)$ for the portfolio is given by [see, e.g., Merton (1969, 1971), Ingersoll (1987), and Duffie (1996)]

$$dW(t) = [rW(t) + A^T(t)h]\,dt + A^T(t)\sigma\,dB_t, \tag{1}$$

where $h \equiv (\mu - r\mathbf{1})$ and where $A^T(t) \equiv (A_1, \ldots, A_N)$ denotes the dollar amount invested in the risky stocks and is a short-hand notation for a function of t, $W(s)$, and $P(s)$, with $s \in [0, t]$.[7] The manager decides how much to invest in bonds and stocks and continuously adjusts his portfolio positions. The conflict may arise because the investor does not observe the manager's portfolio policy vector $A(t)$.

We here focus upon the incentive conflict between an investor and a manager where there is no asymmetric information about stock returns. Justifications for the employment of a portfolio manager may include the manager's lower transaction costs on stocks, the investor's desire for a diversified portfolio, and the investor's lack of time for active investment. Though the belief that a manager may possess superior information is an important reason for his employment, the finding that actively managed mutual funds, on average, underperform passive index funds may raise questions on the validity of this belief.[8] In addition, a manager's information about a limited number of individual stocks will not play a major role if an investor is only interested in asset allocations among a money market fund, a diversified domestic index portfolio, and diversified index portfolios of foreign countries.

If the investor observes both the stock price vector $P(t)$ and the wealth process $W(t)$ of the portfolio continuously, then she can infer precisely the manager's portfolio policy vector $A(t)$ from the fact that the instantaneous covariance between $W(t)$ and $P(t)$ equals $\mathrm{diag}(P)\sigma\sigma^T A(t)$. Since $\sigma\sigma^T$ is invertible by assumption, the manager's policy vector $A(t)$ is completely determined.[9] Hence we must assume that the investor does not observe the wealth and the stock price processes simultaneously. In practice, investors do not know a mutual or pension fund's minute-by-minute value and can only observe the value of the fund at the end of the day. On the other hand, it is relatively easier to track down stock prices. Therefore we assume that

[7] Technically, an admissible $A(t)$ must be progressively measurable with respect to $\mathcal{F}_t$ and must satisfy $E\int_0^T |A(t)|^2\,dt < \infty$, a.s.

[8] The majority of studies now find that actively managed funds, on average, provide lower net returns than passively managed indexes. See, for example, Lakonishok, Shleifer, and Vishny (1992), Elton et al. (1993), Gruber (1996), and Carhart (1997). On the other hand, Grinblatt and Titman (1989, 1993) and Wermers (2000) find that though actively managed mutual funds underperform the market indexes on a net return basis, they may outperform their benchmarks before expenses are deducted. Asymmetric information may play a more prominent role in the case of hedge funds. Recent theoretical work by Dybvig, Farnsworth, and Carpenter (2000), Garcia (2000b), and Sung (2000) may provide insight into the understanding of a delegated portfolio problem in the presence of asymmetric information about stock returns.

[9] We are grateful to the referee for this point.

the investor observes the stock price processes continuously over time and that she observes the terminal value of the portfolio only.[10] Consequently the contract may depend upon the values of $\{t, P(t)\}$, $t \in [0, T]$ and $W(T)$. Throughout this article we assume that the agent's preference over wealth is described by $U_a = -\frac{1}{R_a}\exp[-R_a W_a(T)]$. Here a signifies the agent, and R_a and $W_a(T)$ denote, respectively, the constant risk aversion coefficient and the terminal wealth for the agent. The principal's utility function is mainly constrained to be negative exponential, but may be of a general form in a special case.

At time 0, the principal offers the agent a contract $S[\{t, P(t)\}, W(T)]$, $t \in [0, T]$. Assume that the contract space $S[\{t, P(t)\}, W(T)]$ is of the following form:

$$\begin{aligned} S[\{t, P(t)\}, W(T)] = \mathscr{E}_T + \int_0^T \alpha(t, W(t), P(t))\, dt \\ + \int_0^T \beta_1(t, W(t), P(t)) dW(t) \\ + \int_0^T \beta_2(t, W(t), P(t))\, dP(t), \end{aligned} \tag{2}$$

where $\mathscr{E}_T$ is a general function of T, $W(T)$, and $P(T)$, and where $\alpha(t, W(t), P(t))$, $\beta_1(t, W(t), P(t))$, and $\beta_2(t, W(t), P(t))$ are unknown coefficients that may depend upon t, $W(t)$, and $P(t)$.[11] Note that β_2 and P denote a row and a column vector, respectively. This contract space covers not only all twice continuously differentiable functions of $W(T)$ and $P(T)$, but also path-dependent functions of $P(t)$ except that α, β_1, and β_2 at time t do not depend upon the past history of $P(t)$. The contract space also includes path-dependent functions of $W(t)$, but we shall exclude from our solution contracts that are functions of $W(t)$. It is clear that this contract form includes a general portfolio of risky assets as a benchmark. For example, if β_2 represents a negative constant vector, then the benchmark portfolio is a passive index.

$S[\{t, P(t)\}, W(T)]$ is payable only at time T. Therefore the principal's terminal wealth is $W(T) - S[\{t, P(t)\}, W(T)]$ and the agent's terminal wealth is $S[\{t, P(t)\}, W(T)]$ minus the costs associated with managing the portfolio. If $S[\{t, P(t)\}, W(T)]$ affords the agent at least his reservation utility, $-\frac{1}{R_a}\exp(-R_a\mathscr{E}_0)$, at time 0, he undertakes the job with the understanding that he may not quit it at any time between 0 and T. Here $\mathscr{E}_0$ denotes the agent's certainty equivalent wealth at time 0.

[10] Unlike the agent's effort in a traditional principal-agent problem, the manager's trading records can be verified ex post. Alternatively we may simply assume that forcing contracts are not feasible or that the investor's contract must induce the manager to choose voluntarily optimal trading strategies.

[11] The first three terms are adapted from Schättler and Sung (1993). We thank the referee for suggesting the last term, which captures the symmetric incentive fee structure allowed by the Amendment.

We are now in a position to formally state the agent's and the principal's maximization problems. Given the contract $S[\{t, P(t)\}, W(T)]$, the agent's problem is to maximize his own expected utility over his terminal wealth:

$$\sup_{\{A(t)\}} E\left[-\frac{1}{R_a}\exp\left\{-R_a\left[S[\{t,P(t)\},W(T)]-\int_0^T c(t,A(t),W(t))\,dt\right]\right\}\right]$$
$$\text{s.t.}\quad dW(t)=[rW(t)+A^T(t)h]\,dt+A^T(t)\sigma\,dB_t, \tag{3}$$

where $c(\cdot)$ denotes the monetary cost rate incurred by the agent.

The principal's objective is to choose an $S[\{t, P(t)\}, W(T)]$ and an $\{A(t)\}$ vector so as to maximize her expected utility over the terminal wealth subject to various constraints:

$$\sup_{S[\{t,P(t)\},W(T)],\{A(t)\}} E[U_p[W(T)-S[\{t,P(t)\},W(T)]]]$$
$$\text{s.t.}\quad dW(t)=[rW(t)+A^T(t)h]\,dt+A^T(t)\sigma\,dB_t,$$
$$\text{s.t.}\quad E\left[-\frac{1}{R_a}\exp\left\{-R_a\left[S[\{t,P(t)\},W(T)]-\int_0^T c(t,A(t),W(t))\,dt\right]\right\}\right]$$
$$\geq -\frac{1}{R_a}\exp(-R_a\mathscr{E}_0), \tag{4}$$

and subject to the agent's incentive compatibility constraint that the principal's optimal policy $\{A^*(t)\}$ must also solve the agent's maximization problem. The inequality represents the agent's participation constraint. It is binding at the principal's optimal solution; otherwise the principal could always lower $S[\{t, P(t)\}, W(T)]$ while still getting the agent to accept the job. The solutions to the maximization problems in Equations (3) and (4) shall be referred to as the second-best solutions. Notice that the two maximization problems are not independent; the principal's contract depends upon the agent's action and the agent's action is based upon the principal's contract. Consequently one must solve these two problems together to achieve optimal results.

3. An Expression for the Optimal Fee Structure

In this section we provide an expression for the optimal or equilibrium fee structure to be denoted by $S(T)$ in terms of the agent's value function, β_1, and β_2, using the agent's dynamic maximization problem and his participation constraint. The optimal fee $S(T)$ represents only the equilibrium amount that the principal pays to the agent if the agent adopts the principal's optimal policy. $S(T)$ may or may not implement the principal's optimal policy. The optimal contract $S[\{t, P(t)\}, W(T)]$, however, must implement the optimal policy. $S[\{t, P(t)\}, W(T)]$ and $S(T)$ are equal in equilibrium. It shall be seen that $S(T)$ satisfies the agent's participation constraint and greatly simplifies the principal's maximization problem.

We begin with the agent's maximization problem. Define a value function process $V(t, W, P) \equiv V(t, W(t), P(t))$ for the agent's problem as[12]

$$
\begin{aligned}
&V(t, W, P)\\
&= \sup_{\{A(u)\}} E_t\left[-\frac{1}{R_a}\exp\left\{-R_a\left[\mathscr{E}_T + \int_t^T \alpha(u, W, P)\,du - \int_t^T c(u, W, A)\,du \right.\right.\right.\\
&\qquad\left.\left.\left. + \int_t^T \beta_1(u, W, P) dW(u) + \int_t^T \beta_2(u, W, P)\,dP(u)\right]\right\}\right]\\
&\equiv E_t\left[-\frac{1}{R_a}\exp\left\{-R_a\left[\mathscr{E}_T + \int_t^T \alpha(u, W, P)\,du \right.\right.\right.\\
&\qquad + \int_t^T \beta_1(u, W, P)(rW + A^{*T}h)\,du + \int_t^T \beta_2(u, W, P)\,\mathrm{diag}(P)\mu\,du\\
&\qquad - \int_t^T c(u, W, A^*)\,du + \int_t^T \beta_1(u, W, P)A^{*T}\sigma\,dB_u\\
&\qquad \left.\left.\left. + \int_t^T \beta_2(u, W, P)\,\mathrm{diag}(P)\sigma\,dB_u\right]\right\}\right],
\end{aligned} \tag{5}
$$

where $\{A^*(u)\}$ denotes the solution to the maximization problem in $V(t, W, P)$. We show in Appendix A that under a regularity condition, satisfying the following Bellman-type equation is both a necessary and a sufficient condition for $A^*(t)$ to be an optimal solution:

$$
\begin{aligned}
0 = \sup_{A(t)}&\left[-V(t,W,P)\left\{R_a[\alpha(t,W,P)+\beta_1(t,W,P)(rW+A^T h)\right.\right.\\
&+\beta_2(t,W,P)\,\mathrm{diag}(P)\mu - c(t,W,A)] - \frac{1}{2}R_a^2[\beta_1(t,W,P)A^T\sigma\\
&\left.+\beta_2(t,W,P)\,\mathrm{diag}(P)\sigma][\beta_1(t,W,P)A^T\sigma+\beta_2(t,W,P)\,\mathrm{diag}(P)\sigma]^T\right\}\\
&\quad +V_t + V_W\{rW + A^T h - R_a[\beta_1(t,W,P)A^T\sigma\\
&+\beta_2(t,W,P)\,\mathrm{diag}(P)\sigma]\sigma^T A\} + \frac{1}{2}V_{WW}A^T\sigma\sigma^T A + V_P^T[\mathrm{diag}(P)\mu\\
&-R_a\beta_1(t,W,P)\,\mathrm{diag}(P)\sigma\sigma^T A - R_a\,\mathrm{diag}(P)\sigma(\mathrm{diag}(P)\sigma)^T\beta_2^T(t,W,P)]\\
&\left.+\frac{1}{2}tr[V_{PP}\,\mathrm{diag}(P)\sigma(\mathrm{diag}(P)\sigma)^T] + V_{WP}^T\,\mathrm{diag}(P)\sigma\sigma^T A\right].
\end{aligned} \tag{6}
$$

In Equation (6) the subscripts denote the relevant partial derivatives of $V(t, W, P)$.

[12] For notational convenience, we shall write W and P rather than $W(t)$ and $P(t)$ in $\alpha(\cdot)$, $\beta(\cdot)$, $c(\cdot)$, $V(\cdot)$, etc.

Using the agent's participation constraint, his Bellman equation, Ito's lemma, and a transformation, we can arrive at an expression for the equilibrium fee, which is presented in the next proposition.

Proposition 1. *The equilibrium fee $S(T)$ is given by*

$$\begin{aligned}
S(T) = {} & \mathscr{E}_0 + \int_0^T c\,dt + \frac{1}{2R_a}\int_0^T \bigg| \left(\frac{V_W}{V} - R_a\beta_1\right) A^T\sigma \\
& + \left(\frac{V_P^T}{V} - R_a\beta_2\right)\mathrm{diag}(P)\sigma \bigg|^2 dt \\
& - \frac{1}{R_a}\int_0^T \left[\left(\frac{V_W}{V} - R_a\beta_1\right) A^T\sigma + \left(\frac{V_P^T}{V} - R_a\beta_2\right)\mathrm{diag}(P)\sigma \right] dB_t \\
\equiv {} & \mathscr{E}_0 + \int_0^T c\,dt + \frac{R_a}{2}\int_0^T |\bar{\beta}_1 A^T\sigma + \bar{\beta}_2\,\mathrm{diag}(P)\sigma|^2\,dt \\
& + \int_0^T \left[\bar{\beta}_1 A^T\sigma + \bar{\beta}_2\,\mathrm{diag}(P)\sigma\right] dB_t, \qquad (7)
\end{aligned}$$

where $\mathscr{E}_0$ denotes the agent's certainty equivalent wealth at time 0, *where $\bar{\beta}_1 \equiv \beta_1 - \frac{V_W}{R_a V}$ and $\bar{\beta}_2 \equiv \beta_2 - \frac{V_P^T}{R_a V}$, and where $|\cdot|^2 \equiv [\cdot][\cdot]^T$.*

Proof. See Appendix B. ■

A subtle point which requires special attention is that the equilibrium fee $S(T)$ obtained above does not induce the agent to adopt the principal's optimal policy $A^*(t)$. If this $S(T)$ is offered to the agent, the agent can always achieve his certainty equivalent wealth $\mathscr{E}_0$ by choosing $A^*(t)$ and can perform better by choosing a policy $A(t)$ as long as $c(t, W, A) < c(t, W, A^*)$ a.s. Given the $S(T)$ in Equation (7), it is easy to verify that the agent's participation constraint is satisfied. This constraint thus drops out of the principal's maximization problem.

In the Holmström–Milgrom (1987) and Schättler–Sung (1993) problems, the agent's action $A(t)$ does not appear in the diffusion term, and both $\bar{\beta}_2$ and the stock price vector $P(t)$ are absent from the contract form. It can be shown that the first-order condition (FOC) of the agent's Bellman equation with respect to $A(t)$ leads to an explicit expression for $\bar{\beta}_1$ in terms of $A(t)$. Consequently the agent's Bellman equation is essentially separated from the principal's problem, which can thus be solved independently. The principal's problem is then to determine the optimal policy $A^*(t)$. In the Sung model, the output process is given by $dW(t) = A dt + \sigma\, dB(t)$, where both $A \equiv A(t, W)$ and $\sigma \equiv \sigma(t, W)$ are one-dimensional. The agent controls A and σ separately, and the principal observes either the whole $W(t)$ process or the terminal value $W(T)$ only.[13] The use of the FOC with respect to A alone eliminates β_1 and V from the expression for the optimal fee. Since β_2 and

[13] Since σ is one-dimensional, if the principal observes the whole $W(t)$ process, then she can infer σ exactly.

the stock price vector $P(t)$ are absent in this model, the optimal fee depends only upon A and σ. In the present portfolio management problem, however, the FOC of the agent's Bellman equation is no longer sufficient to provide an expression for $\bar{\beta}_1$ and $\bar{\beta}_2$ in terms of $A(t)$. Therefore we must solve for $\bar{\beta}_1$, $\bar{\beta}_2$, and $V(t, W, P)$ explicitly in order to arrive at an optimal contract $S[\{t, P(t)\}, W(T)]$ that implements $A^*(t)$. In the next two sections we solve for the optimal contracts for two types of problems.

4. An Optimal Contract in Closed Form

The investor's objective is to solve for an optimal contract that implements her optimal policy. Given the equilibrium fee in Equation (7) that satisfies the manager's participation constraint, our strategy is to solve for $A(t)$, $\bar{\beta}_1$, and $\bar{\beta}_2$ so as to maximize the investor's expected utility. Given $A^*(t)$, $\bar{\beta}_1^*$, and $\bar{\beta}_2^*$, we shall then construct an optimal contract in terms of $\{t, P(t)\}$ and $W(T)$ from Equation (7). The point is that the optimal contract $S[\{t, P(t)\}, W(T)]$ must not only solve the manager's dynamic maximization problem, but also reduce to Equation (7) in equilibrium when $A^*(t)$ is indeed the optimal policy. If there exists a set of solutions, then the resulting contract must be optimal for our problem.[14]

For tractability, we assume that the investor's utility function is of the negative exponential form. To solve the investor's problem, we define her value function $J(t, W(t), P(t))$ from the $S(T)$ in Equation (7) as

$$\begin{aligned} J(t, W(t), P(t)) = \sup_{A,\bar{\beta}_1,\bar{\beta}_2} E_t\Bigg\{-\frac{1}{R_p}\exp\Bigg[-R_p\Big[W(T)-\mathscr{C}_0-\int_t^T c\,du \\ -\frac{R_a}{2}\int_t^T |\bar{\beta}_1 A^T\sigma+\bar{\beta}_2\,\text{diag}(P)\sigma|^2\,du \\ -\int_t^T \big[\bar{\beta}_1 A^T\sigma+\bar{\beta}_2\,\text{diag}(P)\sigma\big]\,dB_u\Big]\Bigg]\Bigg\}. \end{aligned}$$

The investor's Bellman equation is then given by

$$\begin{aligned} 0 = \sup_{A,\bar{\beta}_1,\bar{\beta}_2}\Bigg\{J(t, W, P)\Bigg[R_p c+\frac{R_p}{2}(R_a+R_p)|\bar{\beta}_1 A^T\sigma+\bar{\beta}_2\,\text{diag}(P)\sigma|^2\Bigg]+J_t \\ +J_W\{rW+A^T h+R_p\big[\bar{\beta}_1 A^T\sigma+\bar{\beta}_2\,\text{diag}(P)\sigma\big]\sigma^T A\}+\frac{1}{2}J_{WW}A^T\sigma\sigma^T A \\ +J_P^T\{\text{diag}(P)\mu+R_p\text{diag}(P)\sigma[\bar{\beta}_1 A^T\sigma+\bar{\beta}_2\,\text{diag}(P)\sigma]^T\} \\ +\frac{1}{2}tr[J_{PP}\,\text{diag}(P)\sigma(\text{diag}(P)\sigma)^T]+J_{WP}^T\,\text{diag}(P)\sigma\sigma^T A\Bigg\}, \end{aligned} \tag{8}$$

[14] Note that this represents the best possible result that the investor can achieve. Solutions may not always exist because the solutions for $A^*(t)$, $\bar{\beta}_1^*$, and $\bar{\beta}_2^*$, which are determined from the investor's maximization problem, may not always satisfy the FOC of the manager's Bellman equation.

with the boundary condition that $J(T, W, P) = -\frac{1}{R_p}\exp(-R_p W(T))$. Without loss of generality, we can ignore the constant term $\mathscr{E}_0$.

We now need a specific cost function to proceed. Assume that the cost function is given by

$$c(t, A, W) = \frac{1}{2}A^T \mathbf{k}(t)A + \gamma W,$$

where $\mathbf{k}$ denotes an $N \times N$ matrix, with elements $k_{ij}(t)$ being functions of time t, and where γ is a constant. Note that this cost function includes $\frac{1}{2}\sum_{i=1}^{N} k_i(t)A_i^2$ as a special case when $\mathbf{k}(t)$ is a diagonal matrix with $\mathbf{k}_{ii}(t) = k_i$ and $\gamma = 0$. This special cost function is widely used in standard moral hazard problems where $A(t)$ denotes the agent's effort vector. Though it is beyond our model, by adopting this tractable cost function we are implicitly assuming that the more the manager invests in the risky assets, the more effort he must expend in acquiring information about the risky assets as well as in monitoring the stock price movements. The manager's marginal effort and cost increase with the level of investment in risky assets.[15] If $k = 0$, then we have $c(t, W) = \gamma W$, that is, the costs are proportional to the size of the fund. This may be interpreted as the manager's operating costs for managing the portfolio. A mutual fund typically reports an operating cost of 0.5% of the total assets under management and charges its investors for this amount in addition to the management fees.[16] In sum, the more the manager invests in risky assets and the higher the value of the manager's portfolio, the higher the cost that he incurs. $c(t, A, W)\,dt$ represents the manager's total costs in the time interval between t and $t + dt$. The next theorem summarizes a key result of the article.

Theorem 1. *Given the above cost function (or with the addition of a constant term), the optimal portfolio policy and the optimal contract are given by*

$$A^*(t) = f_1(t)\left[\mathbf{k}(t) + \frac{R_a R_p}{R_a + R_p} f_1^2(t)\sigma\sigma^T\right]^{-1} h;$$

$$f_1(t) = \left(1 - \frac{\gamma}{r}\right)e^{r(T-t)} + \frac{\gamma}{r} \tag{9}$$

[15] In a typical moral hazard problem where the agent influences the drift rate or the mean of an output, if the agent is given a fixed payment, then he will not exert any costly effort. An incentive contract is to induce the agent to expend the right effort level. In our delegated portfolio problem, the goal of an incentive contract is to induce the manager to invest the right amount in the risky assets. If the manager is given a fixed compensation, then he will invest all the funds in the riskless asset in order to minimize costs, since investing in the riskless asset requires no effort in information acquisition and other active management activities. The inclusion of a time-dependent function $k_i(t)$ in the cost function is mainly for generality and implies that the cost for the manager's effort associated with managing the portfolio may change over time. We shall show that the form of our optimal contracts does not depend critically upon the form of $k_i(t)$.

[16] As shown in Proposition 1, the investor reimburses the manager his total costs as part of the manager's compensation scheme.

and

$$S[\{t, P(t)\}, W(T)] = F + \frac{R_p}{R_a + R_p} W(T) + \frac{R_a}{R_a + R_p} \times \left[W(T) - \int_0^T f_1(t) A^{*T}(t) [\mathrm{diag}(P)]^{-1} \, dP(t) \right], \quad (10)$$

respectively, where F denotes a constant.

Proof Overview. The detailed proof is given in Appendix C. In summary, we can solve for the optimal A, $\bar{\beta}_1$, and $\bar{\beta}_2$ from the investor's Bellman equation and its three FOCs. In the absence of the FOC of the manager's maximization problem, there are multiple solutions for $\bar{\beta}_1$ and $\bar{\beta}_2$, but the optimal portfolio policy $A^*(t)$ is uniquely determined. Taking the manager's FOC into account, we then find the set of solutions for $\bar{\beta}_1$ and $\bar{\beta}_2$ which guarantees that the resulting contract implements the investor's optimal policy. ■

Given the above optimal contract $S[\{t, P(t)\}, W(T)]$, the manager adopts the investor's optimal portfolio policy $A^*(t)$, and both the manager and the investor share the risk associated with the stock prices or the Brownian motion. Given that the manager adopts $A^*(t)$ in equilibrium, the next corollary presents the optimal fee $S(T)$ in a different form.

Corollary 1. *The optimal fee $S(T)$ can be expressed as*

$$S(T) = F' + \frac{R_p}{R_a + R_p} W(T) + \left(\frac{R_a}{R_a + R_p} \right) \frac{\gamma}{r} \left[W(T) - \int_0^T A^{*T}(t) [\mathrm{diag}(P)]^{-1} \, dP(t) \right], \quad (11)$$

where F' denotes a constant. If $\mathbf{k}(t) = 0$ or $c(t, A, W) = \gamma W(t)$, then $S(T)$ can also be adopted as an optimal contract, implementing the investor's optimal policy $A^*(t)$.

Proof. See Appendix C. ■

Remark 1. The optimal contract or fee is of a symmetric form, with the benchmark being a portfolio of risky assets. $f_1(t) A^{*T}(t)[\mathrm{diag}(P)]^{-1}$ in Equation (10) or $A^{*T}(t)[\mathrm{diag}(P)]^{-1}$ in Equation (11) can be interpreted as the number of shares invested in the risky assets in the benchmark portfolio. Unlike a passive index benchmark portfolio in which the number of shares in each risky asset is typically fixed, our benchmark portfolio invests a time-dependent and stochastic number of shares in each risky asset. According to the optimal contract or fee, the investor should pay the manager a fixed fee, a fraction of the total assets under management, plus a bonus or a penalty depending upon the excess return between the managed portfolio and the benchmark portfolio. Our contract not only provides a possible theoretical support for the existing regulation that restricts the incentive fee paid to a mutual or pension fund to be of the symmetric form, but also offers an easily enforceable benchmark against which the fund's performance should be measured. Furthermore, adding a constant to the cost function would only add a constant term to the optimal contract or fee. Everything else would remain intact.

Remark 2. This delegated portfolio management problem can also be interpreted as a standard principal-agent problem in which the $dW(t)$ process denotes the output process for a project and the agent's action influences both the drift and the diffusion terms simultaneously. Our model therefore represents an extension of the Holmström–Milgrom (1987) and Schättler–Sung (1993) models in which the agent controls the drift only. The $dP(t)$ vector process may be interpreted as additional signals observable to both the principal and the agent.

Remark 3. The optimal portfolio policy is a deterministic function of time t. As Holmström and Milgrom (1987) point out that optimal actions and optimal contracts in a one-period principal-agent relationship are typically very complicated. Our optimal contract given in Equation (10) is quite simple and holds regardless of the functional form of $\mathbf{k}(t)$. The cost can even be a path-dependent function of $W(t)$, though the optimal contract must be path independent. Since the contract depends on the intertemporal stock prices, observing the stock prices continuously does add value to the principal.

Remark 4. Recall that the equilibrium fee $S(T)$ represents the equilibrium amount that the investor pays to the manager if the manager adopts the optimal policy $A^*(t)$. Given $A^*(t)$, $S(T)$ is equivalent to the optimal contract $S[\{t, P(t)\}, W(T)]$. $S(T)$ may or may not implement $A^*(t)$. However, when $c(t, A, W) = \gamma W(t)$, Corollary 1 shows that $S(T)$ also implements $A^*(t)$. In other words, our problem has two equivalent solutions $S(T)$ and $S[\{t, P(t)\}, W(T)]$. According to Theorem 1, we can always use a benchmark portfolio in the manager's compensation scheme regardless of the cost function.

When the cost function is independent of the portfolio policy vector $A(t)$, Corollary 1 presents an alternative contract in which the cost function plays an important role. When the cost is a constant, the optimal contract reduces to a linear sharing rule. This is, of course, not in violation of the Amendment, since the rule requires the incentive compensation to be of a symmetric form only when a benchmark is adopted. When the cost function is proportional to the size of the fund, the symmetric incentive performance fee provides efficient risk sharing about the stochastic cost function between the investor and the manager as well as incentives for the manager to adopt the optimal portfolio policy $A^*(t)$. Since the manager is risk averse, he has an incentive to follow the benchmark portfolio $A^*(t)$ given an incentive performance fee. If the manager adopts $A^*(t)$, then it can be shown that the incentive performance fee or the last term in $S(T)$ is equal to $-\frac{R_p}{R_a+R_p}\gamma\int_0^T W(t)\,dt$ plus a constant, which allows the manager and investor to share the risk associated with the path-dependent cost function efficiently.

Since the issue regarding optimal contracts and appropriate benchmarks for fund managers is of great interest and has not been addressed in a general principal-agent framework in the literature, we next present another optimal contract in closed form using more general utility functions for the investor.

5. An Additional Optimal Contract in Closed Form

In this section we solve the problem using more general utility functions for the investor. Equation (7) for the manager's optimal fee is valid regardless of the investor's preferences. When the investor's preference is of a nonexponential form, the investor's dynamic maximization problem becomes analytically intractable in the presence of a nontrivial cost function. To gain insight into our model, we next solve a special case in which the manager's cost function is a constant and the markets are complete, that is, $d = N$. The purpose of this section is to demonstrate with more examples that the optimal contracts are of the symmetric form and that a portfolio of risky assets can indeed be adopted as an appropriate benchmark within the framework considered in this article. It shall be seen that it is still interesting and nontrivial to solve for robust optimal contracts in the second-best case even with a constant cost function. We first consider the first-best solutions and then show that the first-best and the second-best solutions coincide in equilibrium.

5.1 The optimal fee in the first-best case

The first-best solutions arise when the investor can force the manager to adopt any specific portfolio policy. As a result, the manager's incentive compatibility constraint drops out of the investor's maximization problem. The investor's problem is thus to choose an optimal fee[17] subject to only the manager's participation constraint and to determine the optimal portfolio policies subject to the budget constraint. The next proposition presents an expression for the optimal fee.

Proposition 2. *The optimal fee $S(T)$ in the first best case is given by*

$$S(T) = \mathscr{E}_0 + \frac{1}{2R_a} y^T y T + \frac{1}{R_a} y^T (B_T - B_0), \tag{12}$$

where the constant vector y is defined as $y \equiv \sigma^{-1} h$.

Proof. See Appendix D. ■

Remark 1. The $S(T)$ in Equation (12), which depends upon the Brownian motion $(B_T - B_0)$, cannot be a limiting case[18] of an optimal contract

[17] The optimal fee is the same as the optimal contract in the first-best situation.

[18] We define the limiting case as the one in which both $\mathbf{k}(t)$ and γ go to zero.

$S[\{t, P(t)\}, W(T)]$ when $c(t, W, A)$ is a general function of $A(t)$ and $W(t)$, and is thus not a robust contract in the second-best solutions. For example, when $\mathbf{k}(t)$ and γ go to zero, the optimal contract in Section 4 does not reduce to this $S(T)$, which represents only the optimal fee in the second-best case. According to Theorem 1 and Corollary 1, when the cost function goes to zero, the optimal contract reduces to either $S[\{t, P(t)\}, W(T)] = F + \frac{R_p}{R_a+R_p} W(T) + \frac{R_a}{R_a+R_p}[W(T) - \int_0^T f_1(t) A^{*T}(t)[\text{diag}(P)]^{-1}\, dP(t)]$ or $S(T) = F' + \frac{R_p}{R_a+R_p} W(T)$. Our linear contract $S(T)$ generalizes the linear risk-sharing rule obtained by Wilson (1968), Ross (1973, 1974), Leland (1978), and others in a one-period setting. It can be shown that at the optimal solution $A^*(t)$, both contracts are equivalent to the first-best fee given in Equation (12).

Remark 2. The optimal fee $S(T)$ in Equation (12) holds for any investor with a smooth utility function, because the investor's utility function is not required for the derivation of Equation (12). In other words, the manager's absolute amount of compensation is always the same regardless of the investor's preferences. Similar calculations can be performed for a manager with a general utility function. For example, if $U_a(\cdot) = \log(\cdot)$, then $S(T)$ is given by $S(T) = \mathscr{E}_0 \exp[y^T(B_T - B_0)]$. In general, we can conclude that in the first-best situation and given a manager's preference, all investors, regardless of the preferences, pay the same amount to the manager.

We next derive an optimal contract in terms of $W(T)$ and $\{t, P(t)\}$ that implements the investor's optimal policy.

5.2 The optimal contract in the second-best case

In the second-best situation where the investor may not force the manager to choose a specific portfolio policy, a robust optimal contract $S[\{t, P(t)\}, W(T)]$ may not be solved from the investor's problem alone as in the first-best solutions. We must take the manager's dynamic maximization problem into account.[19] The next theorem presents an optimal contract that implements the investor's optimal policy and that reduces to the first-best fee in equilibrium.

Theorem 2. *When the investor's utility function is given by* $U_p[W(T) - S(\cdot)] = \frac{1}{b}[W(T) - S(\cdot)]^b$, $b < 1$, *the optimal portfolio policy is given by*

$$A^{*T}(t) = e^{rt}\left\{\frac{e^{-rT}}{R_a} + \frac{1}{1-b}\left[W(0) - e^{-rT}\left(\mathscr{E}_0 - \frac{1}{2R_a} y^T y T\right)\right]\right.$$
$$\left.\times \exp\left[\frac{1}{1-b} y^T B_t + \frac{1-2b}{2(1-b)^2} y^T y t\right]\right\} y^T \sigma^{-1}$$

[19] Specifically, given $S[\{t, P(t)\}, W(T)]$, the investor's optimal portfolio policy vector $A^*(t)$ must satisfy the manager's Bellman equation and its FOC. In addition, it should be verified that the regularity condition given in Appendix A is satisfied.

$$= e^{rt}\left\{\frac{e^{-rt}}{1-b}W(t)+\frac{e^{-rT}}{R_a}-\frac{e^{-rT}}{1-b}\left[\mathscr{E}_0-\frac{1}{2R_a}y^TyT\right.\right.$$
$$\left.\left.+\frac{1}{R_a}y^Tyt+\frac{1}{R_a}y^TB_t\right]\right\}y^T\sigma^{-1}, \tag{13}$$

where $W(t)$ and $y^T(B_t-B_0)$ or the price vector $P(t)$ have a one-to-one correspondence through Equation (14):

$$e^{-rt}W(t)=e^{-rT}\left[\mathscr{E}_0-\frac{1}{2R_a}y^TyT+\frac{1}{R_a}y^Tyt+\frac{1}{R_a}y^T(B_t-B_0)\right]$$
$$+\left[W(0)-e^{-rT}\left(\mathscr{E}_0-\frac{1}{2R_a}y^TyT\right)\right]$$
$$\times\exp\left[\frac{1}{1-b}y^TB_t+\frac{1-2b}{2(1-b)^2}y^Tyt\right]. \tag{14}$$

The optimal contract that implements the policy in Equation (13) *is given by*

$$S[\{t,P(t)\},W(T)]=F+\int_0^T\left[ne^{r(T-t)}rA^{*T}(t)\mathbf{1}-\frac{1}{R_a}y^T\sigma^{-1}\mu\right]dt+n_1W(T)$$
$$+n_2\left[W(T)-\int_0^T\frac{1}{n_2}\left[ne^{r(T-t)}A^{*T}(t)-\frac{1}{R_a}y^T\sigma^{-1}\right]\right.$$
$$\left.\times[\mathrm{diag}(P)]^{-1}\,dP(t)\right], \tag{15}$$

where F denotes a constant and where n_1, n_2, and n are arbitrary positive constants, with $n=n_1+n_2\le 1$.

Proof Overview. The detailed proof is given in Appendix D. Conjecture that

$$\frac{1}{R_a}y^T=\bar{\beta}_1A^{*T}(t)\sigma+\bar{\beta}_2\,\mathrm{diag}(P)\sigma.$$

Substituting the above relation into Equation (7) immediately yields

$$S(T)=\mathscr{E}_0+\frac{1}{2R_a}y^TyT+\frac{1}{R_a}y^T(B_T-B_0).$$

This means that if we can find an optimal contract based on a set of solutions for $\bar{\beta}_1$ and $\bar{\beta}_2$, then the second-best contract shall reduce to the first-best one in equilibrium. Appendix D shows that if

$$\bar{\beta}_1=ne^{r(T-t)}\quad\text{and}\quad\bar{\beta}_2=\left[\frac{1}{R_a}y^T\sigma^{-1}-ne^{r(T-t)}A^{*T}(t)\right][\mathrm{diag}(P)]^{-1},$$

then we can arrive at the optimal contract given in Equation (15) by rewriting the $S(T)$ in Equation (7) in terms of $W(T)$ and $\{t,P(t)\}$. Appendix D then shows that the resulting contract implements the investor's optimal policy $A^*(t)$ or that given the above $S[\{t, P(t)\}, W(T)]$, $A^*(t)$ solves the manager's dynamic maximization problem. ■

Remark 1. This optimal contract is symmetric. According to this contract, the investor should pay the manager a fixed fee, a stochastic amount involving the stock prices, a fraction of the total assets under management, plus a bonus or a penalty depending upon the portfolio's excess return relative to a benchmark portfolio. The appropriate benchmark is once again a portfolio of risky assets rather than a passive index. In this benchmark portfolio, the number of shares invested in each stock varies with time and the stock price.

Remark 2. It is well known [Merton (1969, 1971)] that in the absence of a manager, an investor with a power utility function invests a constant fraction of her wealth in the risky stocks. It has been shown here that in the presence of a manager, the portfolio policies are quite different: the amount invested in the risky stocks can be more or less than the constant fraction of her wealth because of the normally distributed $(B_t - B_0)$ term in Equation (13). Since in the complete markets there is a one-to-one correspondence between the Brownian motion B_t and the stock prices $P(t)$, $A^*(t)$ can also be expressed in terms of $P(t)$.

6. Conclusion

This article provides new solutions to the contracting problem between an individual investor and a professional portfolio manager. Various optimal contracts are obtained in closed form by considering both the manager's and the investor's dynamic maximization problems and their first-order conditions. Our optimal contracts do employ a benchmark portfolio and are indeed symmetric. They suggest that the optimal benchmark is a portfolio of risky assets in which the number of shares invested in each asset varies over time, rather than a passive index. While our contribution has extended the understanding of optimal contracts between investors and fund managers and seems to support a policy of symmetric contracts, to draw more conclusive policy implications may require extensions of our model to more general settings. For example, while our model requires the investor to invest all of her wealth with the manager, new results may emerge when the investor entrusts only part of her wealth to the manager and invests the rest in an index fund, which incurs relatively little cost.[20] Further implications may also arise when many investors are allowed to interact with many managers. In addition, new implications may arise when it is recognized that the mutual or pension fund

[20] Bizer and DeMarzo (1999) consider optimal contracts when agents can save, borrow, and default.

problem involves vertically related principal-agent relations, that is, the relation between individual investors and the fund company and that between the fund company and the manager who actively manages the portfolio.[21]

Appendix A: Proof of and Verification Result for the Agent's Bellman Equation

Since the partial derivatives of the agent's value function with respect to the stock prices do not play any role in determining optimal contracts and optimal policies in the present article, we shall ignore them in this appendix for simplicity of presentation. We first show that $A^*(t)$ must satisfy the Bellman Equation [Equation (6)]. By definition, $\{A^*(u)\}$ solves the following maximization problem of

$$V(t,W) = E_t\left[-\frac{1}{R_a}\exp\left\{-R_a\left[\mathscr{E}_T + \int_t^T [\alpha + \beta_1(rW + A^{*T}h) + \beta_2\,\mathrm{diag}(P)\mu - c]\,du \right.\right.\right.$$
$$\left.\left.\left. + \int_t^T [\beta_1 A^{*T}\sigma + \beta_2\,\mathrm{diag}(P)\sigma]\,dB_u\right]\right\}\right] \quad \forall\ t. \tag{16}$$

Multiplying both sides of the above equation by $\exp\{-R_a[\int_0^t(\cdot)\,du + \int_0^t(\cdot)\,dB_u]\}$ gives

$$\left[\exp\left\{-R_a\left[\int_0^t [\alpha + \beta_1(rW + A^{*T}h) + \beta_2\,\mathrm{diag}(P)\mu - c]\,du \right.\right.\right.$$
$$\left.\left.\left. + \int_0^t [\beta_1 A^{*T}\sigma + \beta_2\,\mathrm{diag}(P)\sigma]\,dB_u\right]\right\}\right]V(t,W)$$
$$= E_t\left[-\frac{1}{R_a}\exp\left\{-R_a\left[\mathscr{E}_T + \int_0^T \big[\alpha + \beta_1(rW + A^{*T}h) + \beta_2\,\mathrm{diag}(P)\mu - c\big]\,dt \right.\right.\right.$$
$$\left.\left.\left. + \int_0^T [\beta_1 A^{*T}\sigma + \beta_2\,\mathrm{diag}(P)\sigma]\,dB_t\right]\right\}\right].$$

Since the conditional expectation is a martingale, the drift of the process on the left-hand side must vanish. Evaluating its drift using Ito's lemma, we find

$$0 = H(t,W,A^*) \equiv \left[-V(t,W)\left\{R_a[\alpha + \beta_1(rW + A^{*T}h) + \beta_2\,\mathrm{diag}(P)\mu - c]\right.\right.$$
$$\left. - \frac{1}{2}R_a^2|\beta_1 A^{*T}\sigma + \beta_2\,\mathrm{diag}(P)\sigma|^2\right\} + V_t + V_W\{rW + A^{*T}h$$
$$\left. - R_a\big[\beta_1 A^{*T}\sigma + \beta_2\,\mathrm{diag}(P)\sigma\big]\sigma^T A^*\} + \frac{1}{2}V_{WW}|A^{*T}\sigma|^2\right]. \tag{17}$$

We now show that under a regularity condition to be specified below, Equation (17) is equivalent to the agent's Bellman equation [Equation (6)]:

$$H(t,W,A^*) = \sup_{A(t)} H(t,W,A) \geq H(t,W,A),$$

[21] Gervais, Lynch, and Musto (1999) realize that a mutual fund's manager does not negotiate directly with its investors, but rather with a board of directors which is paid to monitor the fund's operation. They develop a two-period model that finds a positive economic role for this extra layer of relationship.

where $A(t)$ is an arbitrary policy for the initial wealth $W(0)$. Let $W^A(t)$ be the associated wealth process solving

$$dW^A(t) = (rW + A^T h)\,dt + A^T \sigma dB_t.$$

Suppose there exists a portfolio policy $A'(t)$ such that $H(t, W, A') > H(t, W, A^*) = 0$ for all t, we can then define a stochastic process $J_t^{A'}$ as

$$J_t^{A'} = \exp\left\{-R_a\left[\int_0^t \alpha(\cdot)\,du - \int_0^t c(\cdot)\,du + \int_0^t \beta_1(\cdot)dW^{A'}(u) + \beta_2(\cdot)\,dP(u)\right]\right\} V(t, W^{A'}(t)),$$

where $V(t, W^{A'}(t))$ is a well-defined value function[22] with the boundary condition of $V(T, W^{A'}) = -\frac{1}{R_a}\exp(-R_a\mathscr{E}_T)$, and where $(\cdot) \equiv (u, W^{A'}, P)$ with the exception that $c(\cdot) \equiv c(u, W, A')$. We assume that in $V(t, W^{A'}(t))$, $A'(s) = A^*(s)$, $\forall\ s > t$, meaning that the control $A'(s)$ in the process $J_t^{A'}$ is switched to the optimal control $A^*(s)$ immediately after time t. At the terminal date T, $J_T^{A'}$ is then given by

$$J_T^{A'} = -\frac{1}{R_a}\exp\left\{-R_a\left[\mathscr{E}_T + \int_0^T \alpha(\cdot)\,dt - \int_0^T c(\cdot)\,dt + \int_0^T \beta_1(\cdot)dW^{A'}(t) + \beta_2(\cdot)\,dP(t)\right]\right\}.$$

Similarly, a stochastic process J_t^A may be defined for an arbitrary policy $A(t)$, including $A^*(t)$. Let $J_0^A \equiv J(0)$ for all $A(t)$, implying that this stochastic process J_t^A starts at the same point at time 0 and may end at different points at time T, depending upon the portfolio policy taken between times 0 and T. A straightforward application of Ito's lemma yields

$$dJ_t^{A'} = F_t^{A'}\left[H(t, W, A')\,dt + \sigma(t, W, A')\,dB_t\right],$$

where $F_t^{A'}$ and $\sigma(t, W, A')$ are given by

$$F_t^{A'} = \exp\left\{-R_a\left[\int_0^t \alpha(\cdot)\,du - \int_0^t c(\cdot)\,du + \int_0^t \beta_1(\cdot)dW^{A'}(u) + \int_0^t \beta_2(\cdot)\,dP(u)\right]\right\}$$

and

$$\sigma(t, W, A') = V_W(t, W^{A'})A'^T\sigma - R_a V(t, W^{A'})\left[\beta_1(\cdot)A'^T\sigma + \beta_2(\cdot)\,\mathrm{diag}(P)\sigma\right],$$

respectively. We thus have

$$J_T^{A'} = J(0) + \int_0^T F_t^{A'} H(t, W, A')\,dt + \int_0^T F_t^{A'}\sigma(t, W, A')\,dB_t > J(0) + \int_0^T F_t^{A'}\sigma(t, W, A')\,dB_t, \tag{18}$$

where we have used the fact that both $F_t^{A'}$ and $H(t, W, A')$ are positive for all t.

Since the agent's utility function is of a negative exponential form, the value function $V(t, W^A)$ or J_t^A cannot be positive. Rearranging the above equation, we have

$$-J_T^{A'} < -J(0) - \int_0^T F_t^{A'}\sigma(t, W, A')\,dB_t.$$

Following an argument in Duffie (1996, Chapter 9): since $-J_T^{A'}$ is nonnegative, a positive process M can be defined by $M_t = -J(0) - \int_0^t F_u^{A'}\sigma(u, W, A')\,dB_u$. We know that M is a local

[22] It is continuously differentiable in t and twice continuously differentiable in $W^{A'}(t)$.

martingale and that a positive (or a bounded from below) local martingale is a super-martingale. By taking the expectations of each side of and rearranging the above equation, we obtain

$$E[J_T^{A'}] = E\left[-\frac{1}{R_a}\exp\left\{-R_a\left[\mathscr{E}_T + \int_0^T \alpha(\cdot)\,dt - \int_0^T c(\cdot)\,dt + \int_0^T \left[\beta_1(\cdot)dW^{A'}(t) + \beta_2(\cdot)\,dP(t)\right]\right]\right\}\right] > J(0). \tag{19}$$

A similar calculation applies with $A(t) = A^*(t)$, yielding

$$J_T^{A^*} = J(0) + \int_0^T F_t^{A^*}\sigma(t, W, A^*)\,dB_t.$$

Assuming that $E[\int_0^T |F_t^{A^*}\sigma(t, W, A^*)|^2\,dt] < \infty$, we know that $E[\int_0^T F_t^{A^*}\sigma(t, W, A^*)\,dB_t] = 0$. [see, e.g., Duffie (1996) and Protter (1990)]. We can then take the expectation of both sides and have

$$E\left[J_T^{A^*}\right] = E\left[-\frac{1}{R_a}\exp\left\{-R_a\left[\mathscr{E}_T + \int_0^T \alpha(\cdot)\,dt - \int_0^T c(\cdot)\,dt + \int_0^T \left[\beta_1(\cdot)dW^{A^*}(t) + \beta_2(\cdot)\,dP(t)\right]\right]\right\}\right] = J(0), \tag{20}$$

where $(\cdot) \equiv (t, W^{A^*}, P)$. Equations (19) and (20) contradict the assumption that $\{A^*(u)\}$ solves the problem in Equation (16). Hence $H(t, W, A')$ cannot be strictly positive for every $t \in [0, T]$. Furthermore, it can be shown that $H(t, W, A')$ may not be strictly positive at any single t. Suppose $H(t', W, A') > 0$ for time t'. We can then construct a new policy A'', which selects $A'(t)$ between t' and $t' + dt'$ and the optimal policy $A^*(t)$ at any other time. Equation (18) would still hold for $J_T^{A''}$, which would lead to $E[J_T^{A''}] > E[J_T^{A^*}]$, contradicting again the assumption that $A^*(t)$ is the optimal policy. Therefore we can conclude that $0 = H(t, W, A^*) \geq H(t, W, A)$ for all t, completing the proof that the Bellman equation is a necessary condition for $A^*(t)$ to be an optimal policy.

We now show that $A^*(t)$, which satisfies the Bellman equation, is indeed an optimal policy. The proof is essentially the same as discussed above. For an arbitrary policy $A(t)$, the Bellman equation states that $H(t, W, A) \leq 0$. Similarly we can define a process J_t^A as

$$J_t^A = \exp\left\{-R_a\left[\int_0^t \alpha(\cdot)\,du - \int_0^t c(\cdot)\,du + \int_0^t [\beta_1(\cdot)dW^A(u) + \beta_2(\cdot)\,dP(u)]\right]\right\}V(t, W^A(t)),$$

with the same boundary condition that $V(T, W^A) = -\frac{1}{R_a}\exp(-R_a\mathscr{E}_T)$ for all A. Again, we assume that in $V(t, W^A)$, $A(s) = A^*(s)\ \forall\ s > t$.[23] From the Bellman equation and Ito's lemma we have

$$J_T^A \leq J(0) + \int_0^T F_t^A\sigma(t, W, A)\,dB_t,$$

where the equality holds at $A = A^*$. Here the super-martingale argument does not apply. Instead, we need to assume that $E[\int_0^T |F_t^A\sigma(t, W, A)|^2\,dt] < \infty$. We can then take the expectations and arrive at $E[J_T^{A^*}] = J(0) \geq E[J_T^A]$, proving that $A^*(t)$ is an optimal policy.

[23] Schättler and Sung (1993) define a different process in which $A(s)$ coincides with $A^*(s)$ prior to t and two controls differ afterwards. In our definition of J_t^A, $A(s)$ differs from $A^*(s)$ prior to and including t and is identical to $A^*(t)$ immediately after t.

Appendix B: Proof of Proposition 1

By Ito's lemma, $dV(t, W, P)$ is given by

$$dV(t,W,P) = \left\{V_t + V_W(rW + A^T h) + \frac{1}{2}V_{WW}A^T\sigma\sigma^T A + V_P^T\,\mathrm{diag}(P)\mu + V_{WP}^T\,\mathrm{diag}(P)\mu\sigma^T A\right.$$
$$\left. + \frac{1}{2}tr[V_{PP}\,\mathrm{diag}(P)\sigma(\mathrm{diag}(P)\sigma)^T]\right\}dt + [V_W A^T\sigma + V_P^T\ \mathrm{diag}(P)\sigma]\,dB_t.$$

From the Bellman equation [Equation (6)], we have

$$V_t + V_W(rW + A^T h) + \frac{1}{2}V_{WW}A^T\sigma\sigma^T A + V_P^T\,\mathrm{diag}(P)\mu + V_{WP}^T\,\mathrm{diag}(P)\sigma\sigma^T A$$
$$+ \frac{1}{2}tr[V_{PP}\,\mathrm{diag}(P)\sigma\,(\mathrm{diag}(P)\sigma)^T]$$
$$= V(t,W,P)\{R_a[\alpha(\cdot) - c(\cdot) + \beta_1(\cdot)(rW + A^T h) + \beta_2(\cdot)\,\mathrm{diag}(P)\mu]$$
$$- \frac{1}{2}R_a^2|\beta_1(\cdot)A^T\sigma + \beta_2(\cdot)\,\mathrm{diag}(P)\sigma|^2\} + V_W R_a[\beta_1 A^T\sigma + \beta_2\,\mathrm{diag}(P)\sigma]\sigma^T A$$
$$+ V_P^T R_a[\beta_1\,\mathrm{diag}(P)\sigma\sigma^T A + \mathrm{diag}(P)\sigma\,(\mathrm{diag}(P)\sigma)^T\beta_2^T],$$

where $|\cdot|^2 \equiv [\cdot][\cdot]^T$. Combining the above two equations yields

$$dV(t,W,P) = \left[V(t,W,P)R_a\left\{\alpha - c + \beta_1(rW + A^T h) + \beta_2\,\mathrm{diag}(P)\mu\right.\right.$$
$$\left. - \frac{1}{2}R_a|\beta_1 A^T\sigma + \beta_2\,\mathrm{diag}(P)\sigma|^2\right\} + V_W R_a[\beta_1 A^T\sigma + \beta_2\,\mathrm{diag}(P)\sigma]\sigma^T A$$
$$\left. + V_P^T R_a[\beta_1\mathrm{diag}(P)\sigma\sigma^T A + \mathrm{diag}(P)\sigma(\mathrm{diag}(P)\sigma)^T\beta_2^T]\right]dt$$
$$+ [V_W A^T\sigma + V_P^T\mathrm{diag}(P)\sigma]\,dB_t, \tag{21}$$

at $A = A^*$.

Following Holmström and Milgrom (1987) and Schättler and Sung (1993), we define an $\mathscr{E}_t$ process as[24]

$$R_a\mathscr{E}_t = -\log[-R_a V(t,W,P)].$$

Using Equation (21) for $dV(t,W,P)$, we obtain

$$R_a d\mathscr{E}_t = -\frac{dV}{V} + \frac{1}{2}\left(\frac{dV}{V}\right)^2 = -\frac{dV}{V} + \frac{1}{2}\left(\frac{V_W}{V}\right)^2 A^T\sigma\sigma^T A\,dt$$
$$= -R_a[\alpha - c + \beta_1(rW + A^T h) + \beta_2\,\mathrm{diag}(P)\mu]\,dt + \frac{1}{2}R_a^2|\beta_1 A^T\sigma + \beta_2\,\mathrm{diag}(P)\sigma|^2\,dt$$
$$- \frac{V_W}{V}R_a[\beta_1 A^T\sigma + \beta_2\,\mathrm{diag}(P)\sigma]\sigma^T A\,dt$$
$$- \frac{V_P^T}{V}R_a[\beta_1\,\mathrm{diag}(P)\sigma\sigma^T A + \mathrm{diag}(P)\sigma\,(\mathrm{diag}(P)\sigma)^T\beta_2^T]\,dt$$
$$- \left[\frac{V_W}{V}A^T\sigma + \frac{V_P^T}{V}\,\mathrm{diag}(P)\sigma\right]dB_t + \frac{1}{2}\left|\frac{V_W}{V}A^T\sigma + \frac{V_P^T}{V}\,\mathrm{diag}(P)\sigma\right|^2 dt.$$

[24] This definition is merely a mathematical manipulation and has no economic interpretation in the current model. $\mathscr{E}_t$ may be any process that starts at $\mathscr{E}_0$ and ends at $\mathscr{E}_T$.

Note that

$$
\begin{aligned}
&[\alpha+\beta_1(rW+A^T h)+\beta_2\,\mathrm{diag}(P)\mu]\,dt\\
&\qquad=\alpha\,dt+\beta_1 dW(t)+\beta_2\,dP(t)-[\beta_1 A^T\sigma+\beta_2\,\mathrm{diag}(P)\sigma]\,dB_t.
\end{aligned}
$$

We thus have

$$
\begin{aligned}
R_a d\mathscr{E}_t &= R_a c dt - R_a[\alpha\,dt+\beta_1 dW(t)+\beta_2\,dP(t)]+R_a[\beta_1 A^T\sigma+\beta_2\,\mathrm{diag}(P)\sigma]\,dB_t\\
&\quad+\frac{1}{2}R_a^2|\beta_1 A^T\sigma+\beta_2\,\mathrm{diag}(P)\sigma|^2\,dt-\frac{V_W}{V}R_a[\beta_1 A^T\sigma+\beta_2\,\mathrm{diag}(P)\sigma]\sigma^T A dt\\
&\quad-\frac{V_P^T}{V}R_a[\beta_1\,\mathrm{diag}(P)\sigma\sigma^T A+\mathrm{diag}(P)\sigma(\mathrm{diag}(P)\sigma)^T\beta_2^T]\,dt\\
&\quad+\frac{1}{2}\left|\frac{V_W}{V}A^T\sigma+\frac{V_P^T}{V}\,\mathrm{diag}(P)\sigma\right|^2 dt-\left[\frac{V_W}{V}A^T\sigma+\frac{V_P^T}{V}\,\mathrm{diag}(P)\sigma\right]dB_t\\
&= R_a c dt - R_a[\alpha\,dt+\beta_1 dW(t)+\beta_2\,dP(t)]+\frac{1}{2}R_a^2|\beta_1 A^T\sigma+\beta_2\,\mathrm{diag}(P)\sigma|^2\,dt\\
&\quad-\frac{V_W}{V}R_a[\beta_1 A^T\sigma+\beta_2\,\mathrm{diag}(P)\sigma]\sigma^T A dt\\
&\quad-\frac{V_P^T}{V}R_a\,\mathrm{diag}(P)\sigma[\beta_1 A^T\sigma+\bar{\beta}_2\,\mathrm{diag}(P)\sigma]^T dt+\frac{1}{2}\left|\frac{V_W}{V}A^T\sigma+\frac{V_P^T}{V}\,\mathrm{diag}(P)\sigma\right|^2 dt\\
&\quad-\left[\frac{V_W}{V}A^T\sigma-R_a\beta_1 A^T\sigma+\frac{V_P^T}{V}\,\mathrm{diag}(P)\sigma-R_a\beta_2\,\mathrm{diag}(P)\sigma\right]dB_t\\
&= R_a c dt - R_a[\alpha\,dt+\beta_1 dW(t)+\beta_2\,dP(t)]+\frac{1}{2}R_a^2|\beta_1 A^T\sigma+\beta_2\,\mathrm{diag}(P)\sigma|^2\,dt\\
&\quad-R_a\left[\frac{V_W}{V}A^T\sigma+\frac{V_P^T}{V}\,\mathrm{diag}(P)\sigma\right][\beta_1 A^T\sigma+\beta_2\,\mathrm{diag}(P)\sigma]^T dt\\
&\quad+\frac{1}{2}\left|\frac{V_W}{V}A^T\sigma+\frac{V_P^T}{V}\,\mathrm{diag}(P)\sigma\right|^2 dt\\
&\quad-\left[\left(\frac{V_W}{V}-R_a\beta_1\right)A^T\sigma+\left(\frac{V_P^T}{V}-R_a\beta_2\right)\mathrm{diag}(P)\sigma\right]dB_t\\
&= -R_a[\alpha\,dt+\beta_1 dW(t)+\beta_2\,dP(t)]\\
&\quad+\frac{1}{2}\left|\left(\frac{V_W}{V}-R_a\beta_1\right)A^T\sigma+\left(\frac{V_P^T}{V}-R_a\beta_2\right)\mathrm{diag}(P)\sigma\right|^2 dt\\
&\quad+R_a c dt-\left[\left(\frac{V_W}{V}-R_a\beta_1\right)A^T\sigma+\left(\frac{V_P^T}{V}-R_a\beta_2\right)\mathrm{diag}(P)\sigma\right]dB_t,
\end{aligned}
$$

at $A=A^*$. Integrating between 0 and T, we get

$$
\begin{aligned}
S(T) &= \mathscr{E}_0+\int_0^T c dt+\frac{1}{2R_a}\int_0^T\left|\left(\frac{V_W}{V}-R_a\beta_1\right)A^T\sigma+\left(\frac{V_P^T}{V}-R_a\beta_2\right)\mathrm{diag}(P)\sigma\right|^2 dt\\
&\quad-\frac{1}{R_a}\int_0^T\left[\left(\frac{V_W}{V}-R_a\beta_1\right)A^T\sigma+\left(\frac{V_P^T}{V}-R_a\beta_2\right)\mathrm{diag}(P)\sigma\right]dB_t\\
&\equiv \mathscr{E}_0+\int_0^T c dt+\frac{R_a}{2}\int_0^T|\bar{\beta}_1 A^T\sigma+\bar{\beta}_2\mathrm{diag}(P)\sigma|^2\,dt+\int_0^T[\bar{\beta}_1 A^T\sigma+\bar{\beta}_2\mathrm{diag}(P)\sigma]dB_t. \qquad \blacksquare
\end{aligned}
$$

Remarks. Note that the value function $V(t, W, P)$ does not represent the agent's expected utility function at t, which is given by

$$\overline{V}(t,W,P) = \sup_{\{A(t)\}} E_t\left[-\frac{1}{R_a}\exp\left\{-R_a\left[S(T)-\int_0^T c(\cdot)\,dt\right]\right\}\right]$$
$$= V(t,W,P)\times\exp\left\{-R_a\left[\int_0^t \alpha(\cdot)\,du+\int_0^t \beta_1(\cdot)\,dW(u)+\int_0^t \beta_2(\cdot)\,dP(u)-\int_0^t c(\cdot)\,du\right]\right\}.$$

Since $V(t, W, P)$ is not a utility function, it may not be increasing in $W(t)$. Of course, the expected utility function must be increasing in $W(t)$.

Given an $S[\{t, P(t)\}, W(T)]$ in Equation (2), the definition of $V(t, W, P)$ may not be unique. For instance, if we define an $\mathscr{E}_t$ process as $\mathscr{E}_t = W(t)$, $S[\{t, P(t)\}, W(T)]$ may be expressed in two equivalent forms:

$$S[\{t,P(t)\},W(T)] = W(T)+\int_0^T \alpha(t,W,P)\,dt+\int_0^T \beta_1(t,W,P)\,dW(t)+\int_0^T \beta_2(t,W,P)\,dP(t)$$
$$= W(0)+\int_0^T dW(t)+\int_0^T \alpha(t,W,P)\,dt+\int_0^T \beta_1(t,W,P)\,dW(t)$$
$$+\int_0^T \beta_2(t,W,P)\,dP(t).$$

Accordingly, two different value functions can be defined as

$$V_1(t, W, P) = \sup_{\{A(u)\}} E_t\left[-\frac{1}{R_a}\exp\left\{-R_a\left[W(T)+\int_t^T \alpha(u, W, P)\,du-\int_t^T c(u, W, A)\,du\right.\right.\right.$$
$$\left.\left.\left.+\int_t^T \beta_1(u, W, P)\,dW(u)+\int_t^T \beta_2(u, W, P)\,dP(u)\right]\right\}\right]$$

and

$$V_2(t, W, P) = \sup_{\{A(u)\}} E_t\left[-\frac{1}{R_a}\exp\left\{-R_a\left[W(0)+\int_t^T dW(u)+\int_t^T \alpha(u, W, P)\,du\right.\right.\right.$$
$$\left.\left.\left.-\int_t^T c(u, W, A)\,du+\int_t^T \beta_1(u, W, P)\,dW(u)+\int_t^T \beta_2(u, W, P)\,dP(u)\right]\right\}\right]$$
$$= -\frac{1}{R_a}\exp\{-R_a[W(0)-W(t)]\}\times V_1(t, W, P),$$

where $V_1(t, W, P) = V_2(t, W, P)$ only at $t = 0$. But the two value functions correspond to the same contract $S[\{t, P(t)\}, W(T)]$ and the expected utility function is unique. In addition, the optimal policy $A^*(t)$ remains the same irrespective of the definition of the value function. A slight extension of Proposition 1 leads to the following corollary.

Corollary B. *If the wealth and the stock price processes take more general forms such as*[25]

$$dW(t) = f(t, W(t), A(t))\,dt+g(t, W(t), A(t))\,dB_t \equiv f(W)\,dt+g(W)\,dB_t$$

and

$$dP(t) = f(P)\,dt+g(P)\,dB_t,$$

[25] We assume that $f(\cdot)$ and $g(\cdot)$ satisfy the necessary regularity conditions and that they represent the relevant vectors or matrices.

then the expression for the equilibrium fee $S(T)$ is given by

$$
\begin{aligned}
S(T) &= \mathscr{E}_0 + \int_0^T c\,dt + \frac{1}{2R_a}\int_0^T \left| \frac{V_W}{V} g(W) - R_a\beta_1 g(W) + \frac{V_P^T}{V} g(P) - R_a\beta_2 g(P) \right|^2 dt \\
&\quad - \frac{1}{R_a}\int_0^T \left[\frac{V_W}{V} g(W) - R_a\beta_1 g(W) + \frac{V_P^T}{V} g(P) - R_a\beta_2 g(P) \right] dB_t \\
&\equiv \mathscr{E}_0 + \int_0^T c\,dt + \frac{R_a}{2}\int_0^T |\bar\beta_1 g(W) + \bar\beta_2 g(P)|^2\, dt + \int_0^T [\bar\beta_1 g(W) + \bar\beta_2 g(P)]\, dB_t. \quad (22)
\end{aligned}
$$

The proof for this corollary is exactly the same as for Proposition 1 and is thus omitted. This representation of $S(T)$ extends those in Holmström and Milgrom (1987) and Schättler and Sung (1993), where $g(t, W, A)$ is independent of $A(t)$ and where the stock price processes are absent from the contract space.

Appendix C: Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1. Recall from Section 4 that the investor's Bellman equation is given by

$$
\begin{aligned}
0 = \sup_{A, \bar\beta_1, \bar\beta_2} \Bigg\{ & J(t, W, P)\left[R_p c + \frac{R_p}{2}(R_a + R_p)\left| \bar\beta_1 A^T \sigma + \bar\beta_2 \operatorname{diag}(P)\sigma \right|^2 \right] + J_t \\
& + J_W\{rW + A^T h + R_p[\bar\beta_1 A^T\sigma + \bar\beta_2 \operatorname{diag}(P)\sigma]\sigma^T A\} + \frac{1}{2} J_{WW} A^T \sigma\sigma^T A \\
& + J_P^T\{\operatorname{diag}(P)\mu + R_p \operatorname{diag}(P)\sigma[\bar\beta_1 A^T\sigma + \bar\beta_2 \operatorname{diag}(P)\sigma]^T\} \\
& + \frac{1}{2} tr[J_{PP} \operatorname{diag}(P)\sigma(\operatorname{diag}(P)\sigma)^T] + J_{WP}^T \operatorname{diag}(P)\sigma\sigma^T A \Bigg\}, \quad (23)
\end{aligned}
$$

with boundary condition that $J(T, W, P) = -\frac{1}{R_p}\exp(-R_p W(T))$.

The FOCs of the investor's Bellman equation with respect to $\bar\beta_1$, $\bar\beta_2$, and $A(t)$ yield

$$
J(t, W, P)(R_a + R_p)A^T\sigma[\bar\beta_1 A^T\sigma + \bar\beta_2 \operatorname{diag}(P)\sigma]^T + J_W A^T\sigma\sigma^T A + J_P^T \operatorname{diag}(P)\sigma\sigma^T A = 0,
$$

$$
\begin{aligned}
& J(t, W, P)(R_a + R_p)\operatorname{diag}(P)\sigma[\bar\beta_1 A^T\sigma + \bar\beta_2 \operatorname{diag}(P)\sigma]^T + J_W \operatorname{diag}(P)\sigma\sigma^T A \\
& \quad + \operatorname{diag}(P)\sigma[\operatorname{diag}(P)\sigma]^T J_P = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& J(t, W, P)R_p\{c_A + (R_a + R_p)\bar\beta_1\sigma[\bar\beta_1 A^T\sigma + \bar\beta_2 \operatorname{diag}(P)\sigma]^T\} + J_W\{h + 2R_p\bar\beta_1\sigma\sigma^T A \\
& \quad + R_p\sigma[\bar\beta_2 \operatorname{diag}(P)\sigma]^T\} + J_{WW}\sigma\sigma^T A + R_p\bar\beta_1\sigma[\operatorname{diag}(P)\sigma]^T J_P + \sigma[\operatorname{diag}(P)\sigma]^T J_{WP} = 0,
\end{aligned}
$$

respectively. Conjecture that $J(t, W(t), P(t))$ takes the following form:

$$
J(t, W(t), P(t)) = -\frac{1}{R_p}\exp\{-R_p[f_1(t)W(t) + f_2(t)P(t) + f_3(t)]\},
$$

with boundary conditions that $f_1(T) = 1$ and $f_2(T) = f_3(T) = 0$. Substituting $J(t, W(t), P(t))$ into and simplifying the above FOCs, we obtain

$$
-(R_a + R_p)A^T\sigma[\bar\beta_1 A^T\sigma + \bar\beta_2 \operatorname{diag}(P)\sigma]^T + f_1(t)R_p A^T\sigma\sigma^T A + R_p A^T\sigma[f_2(t)\operatorname{diag}(P)\sigma]^T = 0,
$$

$$- (R_a + R_p)\,\mathrm{diag}(P)\sigma[\bar\beta_1 A^T\sigma + \bar\beta_2\,\mathrm{diag}(P)\sigma]^T + f_1(t)R_p\,\mathrm{diag}(P)\sigma\sigma^T A$$
$$+ R_p\,\mathrm{diag}(P)\sigma[f_2(t)\ \mathrm{diag}(P)\sigma]^T = 0,$$

and

$$-c_A - (R_a+R_p)\bar\beta_1\sigma[\bar\beta_1 A^T\sigma + \bar\beta_2\mathrm{diag}(P)\sigma]^T + f_1(t)\{h + 2R_p\bar\beta_1\sigma\sigma^T A + R_p\sigma[\bar\beta_2\mathrm{diag}(P)\sigma]^T\}$$
$$-R_p f_1^2(t)\sigma\sigma^T A + R_p\bar\beta_1\sigma[f_2(t)\mathrm{diag}(P)\sigma]^T - R_p f_1(t)\sigma[f_2(t)\mathrm{diag}(P)\sigma]^T = 0.$$

In order to arrive at a set of solutions that satisfy these three FOCs as well as the investor's Bellman equation, it is necessary that $f_2(t) = 0$. In addition, if

$$[\bar\beta_1 A^T\sigma + \bar\beta_2\,\mathrm{diag}(P)\sigma]^T = \frac{R_p}{R_a + R_p} f_1(t)\sigma^T A, \tag{24}$$

then the FOCs with respect to $\bar\beta_1$ and $\bar\beta_2$ are satisfied. Substituting Equation (24) into the FOC with respect to A, we have

$$\begin{aligned}
0 &= -c_A + f_1(t)h - \bar\beta_1 R_p f_1(t)\sigma\sigma^T A + f_1(t)2R_p\bar\beta_1\sigma\sigma^T A \\
&\quad + f_1(t)R_p\sigma[\bar\beta_2\ \mathrm{diag}(P)\sigma]^T - R_p f_1^2(t)\sigma\sigma^T A \\
&= -c_A + f_1(t)h + R_p f_1(t)\bar\beta_1\sigma\sigma^T A + R_p f_1(t)\sigma[\bar\beta_2\,\mathrm{diag}(P)\sigma]^T - R_p f_1^2(t)\sigma\sigma^T A \\
&= -c_A + f_1(t)h + R_p f_1(t)\sigma[\bar\beta_1 A^T\sigma + \bar\beta_2\,\mathrm{diag}(P)\sigma]^T - R_p f_1^2(t)\sigma\sigma^T A \\
&= -c_A + f_1(t)h + R_p f_1(t)\sigma\frac{R_p}{R_a+R_p} f_1(t)\sigma^T A - R_p f_1^2(t)\sigma\sigma^T A \\
&= -c_A + f_1(t)h - f_1^2(t)\frac{R_a R_p}{R_a + R_p}\sigma\sigma^T A.
\end{aligned}$$

The optimal policy vector is thus given by

$$A^*(t) = f_1(t)\left[\mathbf{k}(t) + \frac{R_a R_p}{R_a + R_p} f_1^2(t)\sigma\sigma^T\right]^{-1} h. \tag{25}$$

Here the $A^*(t)$ vector is uniquely determined, independent of the solutions for $\bar\beta_1$ and $\bar\beta_2$.

Substituting $J(t, W(t), P(t))$ into and simplifying the investor's Bellman equation, we have

$$\begin{aligned}
0 = &-\left[\frac{1}{2}A^T(t)\mathbf{k}(t)A(t) + \gamma W(t) + \frac{R_p}{2} f_1(t)A^T(t)\sigma\sigma^T A(t)\right] \\
&+ f_1(t)\left[rW(t) + A^T(t)h + \frac{R_p}{R_a + R_p} f_1(t)A^T(t)\sigma\sigma^T A(t)\right] \\
&- \frac{1}{2}R_p f_1^2(t)A^T(t)\sigma\sigma^T A(t) + f_3'(t) + f_1'(t)W(t).
\end{aligned} \tag{26}$$

To eliminate the $W(t)$ terms in the above equation we must have

$$-\gamma + rf_1(t) + f_1'(t) = 0; \quad f_1(T) = 1.$$

We thus obtain that

$$f_1(t) = \left(1 - \frac{\gamma}{r}\right)e^{r(T-t)} + \frac{\gamma}{r}.$$

Note that $A^*(t)$ is independent of $W(t)$, and as a result, a proper choice of $f_3(t)$ will then ensure the satisfaction of the investor's Bellman equation.

We now discuss the solutions for $\bar{\beta}_1$ and $\bar{\beta}_2$. For the optimal contract to be both path independent and implementable, we arrive at a set of solutions

$$\bar{\beta}_1 = f_1(t) = \left(1 - \frac{\gamma}{r}\right) e^{r(T-t)} + \frac{\gamma}{r}$$

and

$$\bar{\beta}_2 = -\frac{R_a}{R_a + R_p} f_1(t) A^{*T}(t)[\mathrm{diag}(P)]^{-1}.$$

Constructing an optimal contract from Equation (7), we arrive at

$$\begin{aligned}
S[\{t, P(t)\}, W(T)] &= \text{constant} + \int_0^T \gamma W(t)\,dt + \int_0^T [\bar{\beta}_1 A^{*T}(t)\sigma + \bar{\beta}_2 \,\mathrm{diag}(P)\sigma]\, dB_t \\
&= \text{constant} + \int_0^T \gamma W(t)\,dt + \int_0^T \left[\left(1 - \frac{\gamma}{r}\right) e^{r(T-t)} + \frac{\gamma}{r}\right][dW(t) - rW(t)\,dt] \\
&\quad + \int_0^T \bar{\beta}_2\, dP(t) \equiv F + W(T) + \int_0^T \bar{\beta}_2\, dP(t) \\
&= F + \frac{R_p}{R_a + R_p} W(T) + \frac{R_a}{R_a + R_p}\left[W(T) - \int_0^T f_1(t) A^{*T}(t)[\mathrm{diag}(P)]^{-1} dP(t)\right],
\end{aligned}$$

which is Equation (10) in Theorem 1. In the above derivation we have used the fact that $A^*(t)$ and $\bar{\beta}_2 P(t)$ are deterministic functions of t only.

We now verify that given this $S[\{t, P(t)\}, W(T)]$, the manager's Bellman equation and its FOC are satisfied. The manager's Bellman equation is given by

$$\begin{aligned}
0 = \sup_{A(t)} \Bigg[& -V(\cdot) R_a \left\{ \bar{\beta}_2 \ \mathrm{diag}(P)\mu - \frac{1}{2} A^T(t)\mathbf{k}(t) A(t) - \gamma W(t) - \frac{1}{2} R_a |\bar{\beta}_2 \,\mathrm{diag}(P)\sigma|^2 \right\} + V_t \\
& + V_W[rW(t) + A^T(t)h - R_a \bar{\beta}_2 \,\mathrm{diag}(P)\sigma\sigma^T A(t)] + \frac{1}{2} V_{WW} A^T(t)\sigma\sigma^T A(t) \Bigg],
\end{aligned}$$

where we have omitted the V_P^T, V_{PP}, and V_{WP}^T terms because they do not play any role in the determination of the manager's optimal policy $A(t)$.

Conjecture that the manager's value function is given by

$$V(t, W(t)) = -\frac{1}{R_a} \exp\{-R_a[b_1(t)W(t) + b_2(t)]\},$$

which is independent of $P(t)$. Here $b_1(t)$ and $b_2(t)$ are functions independent of $W(t)$ with boundary conditions that $b_1(T) = 1$ and $b_2(T) = 0$. Substituting $V(t, W)$ into and simplifying the manager's Bellman equation, we have

$$\begin{aligned}
0 = \sup_{A(t)} \Bigg[& \bar{\beta}_2 \,\mathrm{diag}(P)\mu - \frac{1}{2} A^T(t)\mathbf{k}(t) A(t) - \gamma W(t) - \frac{1}{2} R_a \left|\bar{\beta}_2 \ \mathrm{diag}(P)\sigma\right|^2 + b_1'(t) W(t) + b_2'(t) \\
& + b_1(t)[rW(t) + A^T(t)h - R_a \bar{\beta}_2 \ \mathrm{diag}(P)\sigma\sigma^T A(t)] - \frac{1}{2} R_a b_1^2(t) A^T(t)\sigma\sigma^T A(t) \Bigg].
\end{aligned}$$

From this equation, it can be shown that

$$b_1(t) = \left(1 - \frac{\gamma}{r}\right) e^{r(T-t)} + \frac{\gamma}{r} \equiv f_1(t).$$

As long as the manager's optimal portfolio policy is deterministic, his Bellman equation will be satisfied with a proper choice of $b_2(t)$.

We next show that the investor's optimal policy $A^*(t)$ satisfies the FOC of the manager's Bellman equation. The FOC yields

$$\begin{aligned}
&-\mathbf{k}(t)A(t)+b_1h-R_ab_1(t)\sigma[\bar{\beta}_2\,\mathrm{diag}(P)\sigma]^T-R_ab_1^2(t)\sigma\sigma^TA(t)\\
&\quad=-\mathbf{k}(t)A(t)+f_1(t)h+\frac{R_a^2}{R_a+R_p}f_1^2(t)\sigma\sigma^TA^*(t)-R_af_1^2(t)\sigma\sigma^TA(t).
\end{aligned}$$

If $A(t)=A^*(t)$, then we have

$$-\mathbf{k}(t)A(t)+f_1(t)h-f_1^2(t)\frac{R_aR_p}{R_a+R_p}\sigma\sigma^TA(t)=0.$$

The manager's FOC is thus satisfied by $A^*(t)$. It can be shown that the regularity condition given in Appendix A is satisfied.[26] Therefore $S[\{t,P(t)\},W(T)]$ and $A^*(t)$ form an optimal solution to the problem. ■

Proof of Corollary 1. Given the optimal policy $A^*(t)$, we have

$$dW(t)=r[W(t)-A^{*T}(t)\mathbf{1}]\,dt+A^{*T}(t)[\mathrm{diag}(P)]^{-1}\,dP(t)$$

or

$$A^{*T}(t)[\mathrm{diag}(P)]^{-1}\,dP(t)=dW(t)-r[W(t)-A^{*T}(t)\mathbf{1}]\,dt.$$

It can be shown that

$$\begin{aligned}
\int_0^T f_1(t)A^{*T}(t)[\mathrm{diag}(P)]^{-1}\,dP(t)&=\text{constant}+\int_0^T f_1(t)[dW(t)-rW(t)\,dt]\\
&=\text{constant}+\left(1-\frac{\gamma}{r}\right)W(T)+\frac{\gamma}{r}\int_0^T[dW(t)-rW(t)\,dt]\\
&=\text{constant}+\left(1-\frac{\gamma}{r}\right)W(T)+\frac{\gamma}{r}\int_0^T A^{*T}(t)[\mathrm{diag}(P)]^{-1}\,dP(t).
\end{aligned}$$

Substituting the above relation into Equation (10) immediately yields

$$\begin{aligned}
S(T)&=F'+\left[\frac{R_p}{R_a+R_p}+\left(\frac{R_a}{R_a+R_p}\right)\frac{\gamma}{r}\right]W(T)-\left(\frac{R_a}{R_a+R_p}\right)\frac{\gamma}{r}\int_0^T A^{*T}(t)[\mathrm{diag}(P)]^{-1}\,dP(t)\\
&=F'+\frac{R_p}{R_a+R_p}W(T)+\left(\frac{R_a}{R_a+R_p}\right)\frac{\gamma}{r}\left[W(T)-\int_0^T A^{*T}(t)[\mathrm{diag}(P)]^{-1}\,dP(t)\right].
\end{aligned}$$

Notice that the optimal fee $S(T)$ may or may not implement the investor's optimal policy $A^*(t)$.

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[26] Notice that $\sigma(t,W,A^*)$ as defined in Appendix A is given by

$$\begin{aligned}
\sigma(t,W,A^*)&=R_aV(t,W)\left[\left(\frac{V_W}{R_aV}-\beta_1\right)A^{*T}(t)\sigma-\beta_2\,\mathrm{diag}(P)\sigma\right]\\
&=-R_aV(t,W)[\bar{\beta}_1A^{*T}(t)\sigma+\bar{\beta}_2\,\mathrm{diag}(P)\sigma]\\
&=-\frac{R_aR_p}{R_a+R_p}f_1(t)A^{*T}(t)\sigma V(t,W).
\end{aligned}$$

From the definition of $F_t^{A^*}$ as given in Appendix A, it can be shown that $F_t^{A^*}\times V(t,W)=-\frac{1}{R_a}\exp\{-R_a[S(T)-\int_0^T c(\cdot)\,dt]\}$. Since $f_1(t)$ and $A^*(t)$ are both bounded, we can show that $E\left[\int_0^T|F_t^{A^*}\sigma(t,W,A^*)|^2\,dt\right]=\int_0^T E|F_t^{A^*}\sigma(t,W,A^*)|^2\,dt<\infty$, using the relation $[\bar{\beta}_1A^{*T}(t)\sigma+\bar{\beta}_2\mathrm{diag}(P)\sigma]^T=\frac{R_p}{R_a+R_p}f_1(t)\sigma^TA^*(t)$ and the expression for $S(T)$, which coincides with $S[\{t,P(t)\},W(T)]$ in equilibrium. Here, the use of Fubini's theorem is justified because the integrand is positive.

We now show that when $\mathbf{k}(t)=0$ or $c(t,A,W)=\gamma W(t)$, the above $S(T)$ implements $A^*(t)$. Given this $S(T)$, the manager defines a value function as

$$V(t,W,P)=-\frac{1}{R_a}E_t\left[\exp\left\{-R_a\left[\frac{R_p}{R_a+R_p}+\left(\frac{R_a}{R_a+R_p}\right)\frac{\gamma}{r}\right]W(T)-\gamma\int_t^T W(u)\,du\right.\right.$$
$$\left.\left.-\left(\frac{R_a}{R_a+R_p}\right)\frac{\gamma}{r}\int_t^T A^{*T}(u)[\mathrm{diag}(P)]^{-1}\,dP(u)\right\}\right],$$

with the boundary condition that $V(T,W(T),P(T))=-\frac{1}{R_a}\exp\{-R_a[(\frac{R_p}{R_a+R_p})+(\frac{R_a}{R_a+R_p})\frac{\gamma}{r}]$ $W(T)\}$. The manager's Bellman equation takes the same form as in the proof of Theorem 1 except that $\bar{\beta}_2$ and $b_1(t)$ are now given by

$$\bar{\beta}_2=-\left(\frac{R_a}{R_a+R_p}\right)\frac{\gamma}{r}A^{*T}(t)[\mathrm{diag}(P)]^{-1};\quad b_1(t)=\frac{R_p}{R_a+R_p}\left(1-\frac{\gamma}{r}\right)\mathrm{e}^{r(T-t)}+\frac{\gamma}{r}.$$

The present form for $b_1(t)$ is chosen so as to eliminate the $W(t)$ terms in the manager's Bellman equation as well as to satisfy the boundary condition that $b_1(T)=\frac{R_p}{R_a+R_p}+(\frac{R_a}{R_a+R_p})\frac{\gamma}{r}$.

When $\mathbf{k}(t)=0$, the FOC of the manager's Bellman equation yields

$$h-R_a\sigma[\bar{\beta}_2\,\mathrm{diag}(P)\sigma]^T-R_ab_1(t)\sigma\sigma^TA(t)$$
$$=h+\left(\frac{R_a^2}{R_a+R_p}\right)\frac{\gamma}{r}\sigma\sigma^TA^*(t)-\left[\frac{R_aR_p}{R_a+R_p}\left(1-\frac{\gamma}{r}\right)\mathrm{e}^{r(T-t)}+\frac{\gamma}{r}R_a\right]\sigma\sigma^TA(t).$$

If $A(t)=A^*(t)$, then we have

$$h-\frac{R_aR_p}{R_a+R_p}\left[\left(1-\frac{\gamma}{r}\right)\mathrm{e}^{r(T-t)}+\frac{\gamma}{r}\right]\sigma\sigma^TA^*(t)=h-\frac{R_aR_p}{R_a+R_p}f_1(t)\sigma\sigma^TA^*(t)=0,$$

which is the same as the FOC of the investor's Bellman equation. The rest of the proof follows from that of Theorem 1. ■

Appendix D: Proofs of Proposition 2 and Theorem 2

In this appendix we first solve for the first-best fee and then show that the first-best results can be implemented using the contract form given in Theorem 2.

Proof of Proposition 2. Since the markets are dynamically complete by assumption, we employ the martingale representation approach developed by Cox and Huang (1989, 1991), Karatzas, Lehoczky, and Shreve (1987), and Pliska (1986).[27] In this approach, one solves a dynamic portfolio problem by separating it into two parts. First, one transforms the dynamic portfolio problem into a static utility maximization problem and solves the static problem to find the optimal wealth process. Then one applies the martingale representation theorems to determine the portfolio policies needed to generate the optimal wealth process.

The investor's problem in the first-best situation is to

$$\sup_{S(T),W(T)}E[U_p[W(T)-S(T)]] \tag{27}$$

[27] For martingale representation approach with incomplete markets, see, for example, He and Pearson (1991) and Karatzas et al. (1991).

$$\text{s.t.} \quad E\left[-\frac{1}{R_a}\exp\{-R_a S(T)\}\right] \geq -\frac{1}{R_a}\exp(-R_a \mathscr{E}_0),$$
$$\text{s.t.} \quad E^{\mathbb{Q}}[W(T)\mathrm{e}^{-rT}] = E[\xi(T)W(T)\mathrm{e}^{-rT}] \leq W(0),$$

where both constraints bind at the optimal solutions $S(T)$ and $W(T)$,[28] and where, without loss of generality, the constant cost function has been omitted. Here, $W(T)$ instead of $A(t)$ appears in the investor's maximization problem, because in the martingale representation approach, choosing the optimal terminal wealth $W(T)$ is equivalent to choosing the optimal portfolio policies. $\mathbb{Q}$ denotes the equivalent martingale measure or the risk-neutral probability,[29] and $\xi(T) \equiv d\mathbb{Q}/dP$ is given by

$$\xi(T) = \exp\left[-\frac{1}{2}y^T yT - y^T(B_T - B_0)\right],$$

where P is the original measure and where $y \equiv \sigma^{-1}h$.

Assume that the investor's utility function U_p is smooth everywhere with respect to both $W(T)$ and $S(T)$.[30] Forming the Lagrangian, we have

$$\sup_{S(T),W(T)} E\left[U_p[W(T) - S(T)] - \frac{\lambda_S}{R_a}\exp[-R_a S(T)] - \lambda_W \xi(T)\mathrm{e}^{-rT}W(T)\right],$$

where λ_S and λ_W denote the Lagrangian multipliers for the constraints with respect to $S(T)$ and $W(T)$, respectively. The FOCs for this point-wise maximization problem with respect to $S(T)$ and $W(T)$ are given by

$$U_p'[W(T) - S(T)] = \lambda_S \exp[-R_a S(T)] \tag{28}$$

and

$$U_p'[W(T) - S(T)] = \lambda_W \xi(T)\mathrm{e}^{-rT}, \tag{29}$$

respectively.[31] The optimal fee $S(T)$ in terms of $(B_T - B_0)$ is an immediate result of the combination of these two FOCs:

$$\lambda_S \exp[-R_a S(T)] = \lambda_W \mathrm{e}^{-rT}\xi(T). \tag{30}$$

Equation (28) determines $S(T)$ in terms of $W(T)$ or vice versa. With the aid of Equation (30), Equation (28) gives $W(T)$ in terms of $(B_T - B_0)$, from which the investor's optimal portfolio policies are determined. From Equation (30), we have

$$\begin{aligned} S(T) &= -\frac{1}{R_a}\log\left[\frac{\lambda_W}{\lambda_S}\mathrm{e}^{-rT}\right] + \frac{1}{2R_a}y^T yT + \frac{1}{R_a}y^T(B_T - B_0) \\ &= \mathscr{E}_0 + \frac{1}{2R_a}y^T yT + \frac{1}{R_a}y^T(B_T - B_0), \end{aligned}$$

where we have used the manager's participation constraint to equate the first equation with the second one, that is, $\mathscr{E}_0 = -\frac{1}{R_a}\log[\frac{\lambda_W}{\lambda_S}\mathrm{e}^{-rT}]$. ■

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[28] For ease of exposition, $S(T)$ and $W(T)$ rather than $S^*(T)$ and $W^*(T)$ are being used to denote the respective optimal solutions.

[29] See Harrison and Kreps (1979) for the former and Cox and Ross (1976) for the latter.

[30] A function is said to be smooth if it is continuous and has a continuous first-order derivative.

[31] Note that $W(T)$ and $S(T)$ are two independent control variables in the first-best solutions. It can be shown that if $U_p[W(T) - S(T)]$ is concave with respect to $W(T) - S(T)$, then it is concave with respect to both $W(T)$ and $S(T)$. Therefore the FOCs are both necessary and sufficient.

Proof of Theorem 2. Our objective is to solve for a contract $S[\{t, P(t)\}, W(T)]$ that both implements the investor's first-best portfolio policies and reduces to the first-best fee $S(T)$. Consequently the second-best results coincide with the first-best ones.

The first-best fee $S(T)$ is given by

$$S(T) = \mathscr{E}_0 + \frac{1}{2R_a} y^T yT + \frac{1}{R_a} y^T (B_T - B_0). \tag{31}$$

Since this $S(T)$ satisfies the manager's participation constraint, the first-best optimal portfolio policies may be solved from the investor's maximization problem, namely,

$$\sup_{W(T)} E\left[\frac{1}{b}[W(T) - S(T)]^b\right] \tag{32}$$

$$\text{s.t.} \quad E^{\mathscr{Q}}[W(T)e^{-rT}] = E[\xi(T)W(T)e^{-rT}] \leq W(0),$$

where $b < 1$. As in the proof of Proposition 2, the equivalent martingale measure is uniquely represented by $\xi(T) \equiv d\mathscr{Q}/dP = \exp[-\frac{1}{2}y^T yT - y^T(B_T - B_0)]$. According to the Girsanov theorem, $B_t^{\mathscr{Q}} \equiv B_t + yt$ is a standard Brownian motion under $\mathscr{Q}$ [see, e.g., Karatzas and Shreve (1991)].

Solving for the FOC of the point-wise static maximization problem in Equation (32) yields the optimal terminal wealth $W(T)$:

$$[W(T) - S(T)]^{b-1} = \lambda \mathrm{e}^{-rT} \exp\left[-\frac{1}{2} y^T yT - y^T (B_T - B_0)\right], \tag{33}$$

where λ is the Lagrangian multiplier. Using the fact that the discounted optimal wealth process, $W(t)\mathrm{e}^{-rt}$, is a martingale under $\mathscr{Q}$, that is, $W(t)\mathrm{e}^{-rt} = E_t^{\mathscr{Q}}[W(T)\mathrm{e}^{-rT}]$, we obtain

$$\begin{aligned}\mathrm{e}^{-rt} W(t) &= E_t^Q[\mathrm{e}^{-rT} W(T)] = \mathrm{e}^{-rT}\left[\mathscr{E}_0 - \frac{1}{2R_a} y^T yT + \frac{1}{R_a} y^T yt + \frac{1}{R_a} y^T (B_t - B_0)\right] \\ &\quad + \lambda^{\frac{1}{b-1}} \exp\left[\frac{b}{1-b} rT + \frac{b}{2(1-b)^2} y^T yT + \frac{1}{1-b} y^T B_t + \frac{1-2b}{2(1-b)^2} y^T yt\right]\end{aligned} \tag{34}$$

or

$$\begin{aligned}d[\mathrm{e}^{-rt} W(t)] &= \left\{\frac{\mathrm{e}^{-rT}}{R_a} + \frac{\lambda^{\frac{1}{b-1}}}{1-b} \exp\left[\frac{b}{1-b} rT + \frac{b}{2(1-b)^2} y^T yT + \frac{1}{1-b} y^T B_t + \frac{1-2b}{2(1-b)^2} y^T yt\right]\right\} \\ &\quad \times y^T \sigma^{-1} (h\,dt + \sigma\, dB_t) \\ &= \left\{\frac{1}{1-b} \mathrm{e}^{-rt} W(t) + \frac{1}{R_a} \mathrm{e}^{-rT} - \frac{1}{1-b} \mathrm{e}^{rT}\left[\mathscr{E}_0 - \frac{1}{2R_a} y^T yT + \frac{1}{R_a} y^T yt + \frac{1}{R_a} y^T B_t\right]\right\} \\ &\quad \times y^T \sigma^{-1} (h\,dt + \sigma\, dB_t).\end{aligned} \tag{35}$$

From Equation (34), we have

$$\lambda^{\frac{1}{b-1}} \exp\left[\frac{b}{1-b} rT + \frac{b}{2(1-b)^2} y^T yT\right] = W(0) - \mathrm{e}^{-rT}\left(\mathscr{E}_0 - \frac{1}{2R_a} y^T yT\right).$$

To obtain $A^*(t)$, we recall that the $dW(t)$ process is given by

$$dW(t) = [rW(t) + A^T h]\, dt + A^T \sigma\, dB_t.$$

Applying Ito's lemma, we obtain that

$$d[\mathrm{e}^{-rt}W(t)] = \mathrm{e}^{-rt}A^T(t)[h\,dt + \sigma\, dB_t]. \tag{36}$$

The unique decomposition of $d[\mathrm{e}^{-rt}W(t)]$ and comparison of Equation (35) with Equation (36) lead to

$$\begin{aligned} A^{*T}(t) &= \mathrm{e}^{rt}\left\{\frac{\mathrm{e}^{-rT}}{R_a} + \left[W(0) - \mathrm{e}^{-rT}\left(\mathscr{E}_0 - \frac{1}{2R_a}y^T yT\right)\right]\right. \\ &\quad \left.\times \exp\left[\frac{1}{1-b}y^T B_t + \frac{1-2b}{2(1-b)^2}y^T yt\right]\right\} \times y^T\sigma^{-1} \\ &= \mathrm{e}^{rt}\left\{\frac{1}{1-b}W(t) + \frac{1}{R_a}\mathrm{e}^{-rT}\right. \\ &\quad \left. - \frac{1}{1-b}\mathrm{e}^{-rT}\left[\mathscr{E}_0 - \frac{1}{2R_a}y^T yT + \frac{1}{R_a}y^T yt + \frac{1}{R_a}y^T B_t\right]\right\} \times y^T\sigma^{-1}, \end{aligned} \tag{37}$$

where the equilibrium $W(t)$ and $y^T(B_t - B_0)$ have a one-to-one correspondence through Equation (34). We can also express $A^*(t)$ in terms of the price vector $P(t)$.

We now construct a contract $S[\{t, P(t)\}, W(T)]$ that implements the above $A^*(t)$. Comparing the contracts given in Equations (31) and (7), we conjecture that

$$\frac{1}{R_a}y^T = \bar{\beta}_1 A^{*T}(t)\sigma + \bar{\beta}_2\,\mathrm{diag}(P)\sigma; \quad y = \sigma^{-1}h. \tag{38}$$

This is, of course, just one of many solutions. But as long as the final contract implements the investor's $A^*(t)$, this solution will be a right one. As in the proof of Theorem 1, the optimal solution for $\bar{\beta}_1(t)$ is given in this case by $\bar{\beta}_1 = n\mathrm{e}^{r(T-t)}$, with n being a constant. We then have

$$\bar{\beta}_2 = \left[\frac{1}{R_a}y^T - n\mathrm{e}^{r(T-t)}A^{*T}(t)\sigma\right][\mathrm{diag}(P)\sigma]^{-1} = \left[\frac{1}{R_a}y^T\sigma^{-1} - n\mathrm{e}^{r(T-t)}A^{*T}(t)\right][\mathrm{diag}(P)]^{-1}. \tag{39}$$

Substituting the solutions for $\bar{\beta}_1$ and $\bar{\beta}_2$ into Equation (7), we arrive at

$$\begin{aligned} S[\{t, P(t)\}, W(T)] &= \mathscr{E}_0 + \frac{1}{2R_a}y^T yT - \int_0^T [n\mathrm{e}^{r(T-t)}A^{*T}(t)h + \bar{\beta}_2\,\mathrm{diag}(P)\mu]dt \\ &\quad + \int_0^T n\mathrm{e}^{r(T-t)}[dW(t) - rW(t)\,dt] + \int_0^T \bar{\beta}_2 dP(t) \\ &\equiv F - \int_0^T [n\mathrm{e}^{r(T-t)}A^{*T}(t)h + \bar{\beta}_2\,\mathrm{diag}(P)\mu]\,dt + nW(T) \\ &\quad - \int_0^T \left[n\mathrm{e}^{r(T-t)}A^{*T}(t) - \frac{1}{R_a}y^T\sigma^{-1}\right][\mathrm{diag}(P)]^{-1}\,dP(t) \\ &= F + \int_0^T \left[n\mathrm{e}^{r(T-t)}rA^{*T}(t)\mathbf{1} - \frac{1}{R_a}y^T\sigma^{-1}\mu\right]dt + n_1 W(T) \\ &\quad + n_2\left[W(T) - \int_0^T \frac{1}{n_2}\left[n\mathrm{e}^{r(T-t)}A^{*T}(t) - \frac{1}{R_a}y^T\sigma^{-1}\right][\mathrm{diag}(P)]^{-1}\,dP(t)\right], \end{aligned}$$

where $n_1 + n_2 = n$. We take n to be less than or equal to 1. Here, $\frac{1}{n_2}[n\mathrm{e}^{r(T-t)}\ A^{*T}(t) - \frac{1}{R_a}y^T\sigma^{-1}][\mathrm{diag}(P)]^{-1}$ can be interpreted as the number of shares invested in risky assets in a benchmark portfolio.

We now verify that the above contract $S[\{t, P(t)\}, W(T)]$ implements $A^*(t)$. Given this contract, the manager's Bellman equation is given by

$$0 = \sup_{A(t)}\Big[-V(\cdot)R_a\Big[\bar{\beta}_2 \operatorname{diag}(P)\mu - \frac{1}{2}R_a|\bar{\beta}_2 \operatorname{diag}(P)\sigma|^2 - n\mathrm{e}^{r(T-t)}A^{*T}(t)h - \bar{\beta}_2 \operatorname{diag}(P)\mu\Big]$$

$$+V_t + V_W\Big[rW + A^T(t)h - R_a\bar{\beta}_2 \operatorname{diag}(P)\sigma\sigma^T A(t)\Big] + \frac{1}{2}V_{WW}A^T(t)\sigma\sigma^T A(t)\Big],$$

where we have omitted the V_P, V_{PP}, and V_{WP} terms because they do not play any role in determining the manager's optimal policy.

Conjecture that the manager's value function is given by

$$V(t, W(t)) = -\frac{1}{R_a}\exp\{-R_a[b_1(t)W(t) + b_2(t)]\},$$

with the boundary conditions that $b_1(T) = n$ and $b_2(t) = 0$, we then have

$$0 = \sup_{A(t)}\Big[-\frac{1}{2}R_a|\bar{\beta}_2 \operatorname{diag}(P)\sigma|^2 - n\mathrm{e}^{r(T-t)}A^{*T}(t)h + b_1'(t)W(t) + b_2'(t)$$

$$+b_1(t)[rW + A^T(t)h - R_a\bar{\beta}_2 \operatorname{diag}(P)\sigma\sigma^T A(t)] - \frac{1}{2}R_a b_1^2(t)A^T(t)\sigma\sigma^T A(t)\Big].$$

From this equation, we obtain $b_1(t) = n\mathrm{e}^{r(T-t)} = \bar{\beta}_1$.[32] The agent's FOC is then given by

$$h - R_a\sigma[\bar{\beta}_2 \operatorname{diag}(P)\sigma]^T - R_a b_1(t)\sigma\sigma^T A(t) = 0.$$

Using $b_1(t) = \bar{\beta}_1$, we have

$$h = R_a\sigma[\bar{\beta}_1 A^T(t)\sigma + \bar{\beta}_2 \operatorname{diag}(P)\sigma]^T \quad \text{or} \quad \frac{1}{R_a}y^T = [\bar{\beta}_1 A^T(t)\sigma + \bar{\beta}_2 \operatorname{diag}(P)\sigma].$$

By construction, the investor's optimal policy $A^*(t)$ satisfies the above manager's FOC.

We now verify that the manager's Bellman equation is satisfied as well, which is not trivial because $A^*(t)$ is a function of $P(t)$.[33] Simplifying the manager's Bellman equation, we have

$$-\frac{1}{2}R_a|\bar{\beta}_2 \operatorname{diag}(P)\sigma|^2 + b_2'(t) - b_1(t)R_a\bar{\beta}_2 \operatorname{diag}(P)\sigma\sigma^T A^*(t) - \frac{1}{2}R_a b_1^2(t)A^{*T}(t)\sigma\sigma^T A^*(t)$$

$$= -\frac{1}{2}R_a\big[|\bar{\beta}_2 \operatorname{diag}(P)\sigma|^2 + 2\bar{\beta}_2 \operatorname{diag}(P)\sigma\bar{\beta}_1\sigma^T A^*(t) + |\bar{\beta}_1 A^{*T}(t)\sigma|^2\big] + b_2'(t)$$

$$= -\frac{1}{2}R_a|\bar{\beta}_1 A^{*T}(t)\sigma + \bar{\beta}_2 \operatorname{diag}(P)\sigma|^2 + b_2'(t) = -\frac{1}{2R_a}y^T y + b_2'(t) = 0,$$

which is indeed satisfied with a choice of $b_2(t)$ given by $b_2(t) = \frac{1}{2R_a}y^T y(t-T)$. Therefore both the manager's Bellman equation and its FOC are satisfied. It can be shown that the regularity condition given in Appendix A is satisfied.[34] ■

[32] This solution of $b_1(t)$ eliminates the $W(t)$ terms from the manager's Bellman equation.

[33] In Theorem 1, $A^*(t)$ is a deterministic function of t, the manager's Bellman equation is always satisfied with a proper choice of $b_2(t)$.

[34] Using $\bar{\beta}_1 A^{*T}(t)\sigma + \bar{\beta}_2 \operatorname{diag}(P)\sigma = \frac{1}{R_a}y^T$, we arrive at $\sigma(t, W, A^*) = -y^T V(t, W)$ and $F_t^{A^*} \times \sigma(t, W, A^*) = y^T\frac{1}{R_a}\exp\{R_a[S(T) - \int_0^T c(\cdot)\,dt]\}$. The rest of the proof is straightforward.

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